Antiperiodic spin-1/2 XXZ quantum chains by separation of variables: Complete spectrum and form factors

G. Niccoli[1] ,

Abstract In this paper we consider the spin 1/2 highest weight representations for the 6-vertex Yang-Baxter algebra on a finite lattice and analyze the integrable quantum models associated to the antiperiodic transfer matrix. For these models, which in the homogeneous limit reproduces the XXZ spin 1/2 quantum chains with antiperiodic boundary conditions, we obtain in the framework of Sklyanin's quantum separation of variables (SOV) the following results: I) The complete characterization of the transfer matrix spectrum (eigenvalues/eigenstates) and the proof of its simplicity. II) The reconstruction of all local operators in terms of Sklyanin's quantum separate variables. III) One determinant formula for the scalar products of *separates states*, the elements of the matrix in the scalar product are sums over the SOV spectrum of the product of the coefficients of the states. IV) The form factors of the local spin operators on the transfer matrix eigenstates by a one determinant formula given by simple modifications of the scalar product formula.

---

[1]YITP, Stony Brook University, New York 11794-3840, USA, niccoli@max2.physics.sunysb.edu

Contents

1 Introduction

The exact and complete solution of the integrable quantum models by the computation of their spectrum and dynamics is a central issue for the mathematical physics as they play an important role in different research areas. The general correspondence existing between these 1-dimensional quantum models[2] and 2-dimensional (exactly solvable) models of classical statistical mechanics[3] gives an important example of this statement. Another important application is to the quantum statistical systems at finite temperature where the integrable Hamiltonians are used to define partition functions and quantum thermal averages[4]. Here, we present an approach to achieve the exact and complete solution of lattice integrable quantum models in the framework of the quantum inverse scattering method (QISM) [8]-[21]. In particular, this approach is addressed to the large class of integrable quantum models whose spectrum[5] (eigenvalues & eigenstates) can be determined by implementing Sklyanin's quantum separation of variables (SOV) method [22]-[24]. Let us comment that the approach here presented can be considered as the generalization to the SOV framework of the Lyon group method[6] [30]-[31] which was instead developed in the algebraic Bethe ansatz (ABA) framework [8]-[9].

This approach has been recently developed in the case of the lattice quantum sine-Gordon model [9, 21] and the τ_2-model [32] associated by QISM to cyclic representations of the 6-vertex Yang-Baxter algebra. In particular, in the papers [33]-[35] the complete SOV spectrum characterization has been constructed for the lattice quantum sine-Gordon model while in the paper[7] [40] it has been derived for the τ_2-model and consequently for the chiral Potts model [41]-[52], by exploiting the well known links between these two models [41]. Finally, in the papers [53, 54] it has been shown how to reconstruct local operators in terms of the quantum separate variables and write in a determinant form the scalar products of separate states[8] and the matrix elements of local operators on transfer matrix eigenstates.

In the present article we develop this approach for quantum models associated by QISM to highest weight representations of the 6-vertex Yang-Baxter algebra. In particular, we consider the representations corresponding to one of the most prototypical lattice integrable quantum model, i.e. the XXZ spin 1/2 quantum chain [55]. It represents one of the best known quantum models under periodic boundary conditions and a very large literature is dedicated to it [56]-[63]. In particular, it was the basic example for the application of the ABA method and then for the implementation of the Lyon group method to compute matrix elements of local operators.

The circumstance [23] interesting for us is that it is enough to change the boundary conditions into antiperiodic ones that the algebraic Bethe ansatz does not work anymore while Sklyanin's quantum separation of variables can be used to analyze the system. Moreover, it is worth remarking that the thermodynamical limits of the periodic and antiperiodic case are naturally expected coinciding and then the XXZ quantum spin 1/2 chain with antiperiodic boundary conditions is a natural prototype for which develop the full program of analysis from the spectrum up to the correlation functions in the SOV framework. As for this model we can take advantage from the known results worked out for the periodic chain in the ABA framework by the Lyon group [64]-[75] and compare them with our

---

[2]See [1, 2] and references therein.

[3]See [3] and reference therein.

[4]That is the thermodynamical Bethe ansatz [4]-[7].

[5] In the QISM formulation the quantum integrable structure (i.e. the complete set of commuting conserved charges) of the model is generated by the transfer matrix, the trace of the monodromy matrix satisfying the Yang-Baxter algebra or generalization of it.

[6]Always in the ABA framework, see also [25]-[26] for the extension of this method to the higher spin quantum chains and [27]-[29] for the generalization to the reflection algebra case.

[7]See also the series of works [36]-[39] for previous analysis by SOV method of the τ_2-model and for some first result concerning the computation of the form factors of local operators in the special case of the generalized Ising model.

[8]See Section 4 for the definition of these states in our current model.

findings.

Let us mention that so far the results concerning the antiperiodic case are restricted to the construction of the Baxter Q-operator[9] [77] and that of the functional separation of variables of Sklyanin for the XXX case [23] extended in [78] to the XXZ case. It is worth pointing out that Sklyanin's separation of variables in its functional version defines representations of the Yang-Baxter algebra on space of symmetric functions and leads only to the representation of the wave functions of the transfer matrix eigenstates. In fact, the explicit construction of the SOV representation as well as of the transfer matrix eigenstates in the original representation space of the quantum spin chain are missing. These are important information in order to achieve the goal to compute matrix elements of the local operators and one of the tasks of the present article is to cover these gaps.

1.1 Motivation for the use of SOV method

In the framework of quantum integrability, there are several methods to analyze the spectral problem as for example the coordinate Bethe ansatz [56], [3] and [79], the Baxter Q-operator method [3], the algebraic Bethe ansatz [8]-[9], the analytic Bethe ansatz [80]-[81]. However, they suffer in general from one or more of the following problems: i) Reduced applicability; i.e. there exist important examples of quantum integrable models to which some of these methods do not apply. ii) Analysis reduced only to the set of eigenvalues; i.e. some of them do not allow for the construction of the eigenstates. iii) Lack of completeness proof; i.e. the completeness of the spectrum description is not assured by the methods but has to be separately proven[10].

The SOV method of Sklyanin is a more promising approach: It works for a large class of integrable quantum models to which ABA does not apply; it leads to both the eigenvalues and the eigenstates of the transfer matrix with a spectrum construction (which under simple conditions) has as built-in feature its completeness. Moreover, for the so far analyzed cases [33]-[35], [40], [83]-[85] in the SOV framework it was an easy task to prove the simplicity of the transfer matrix or to add to it commuting operators which form a complete[11] set of commuting conserved charges of the quantum model.

2 Antiperiodic 6-vertex quantum integrable chain

2.1 Representations on spin-1/2 chain of Yang-Baxter algebra

Let us define a class of representations of the 6-vertex Yang-Baxter algebra on spin-1/2 quantum chains. More in details, let us denote with $\sigma_n^\pm$ and σ_n^z the generators of N independent (local) $sl(2)$ algebras:

$$[\sigma_n^z, \sigma_m^\pm] = \pm\delta_{n,m}\sigma_n^\pm, \quad [\sigma_n^+, \sigma_m^-] = 2\delta_{n,m}\sigma_n^z, \tag{2.1}$$

and let us introduce the 2-dimensional linear spaces (local quantum spaces of the chain) $\mathsf{R}_n \simeq \mathbb{C}^2$. In any linear space R_n is define a spin-1/2 representation of the $sl(2)$ algebra where the generators of the algebra admit the standard representation in terms of 2×2 Pauli matrices. Then, for any local quantum space R_n with $n \in \{1, ..., \mathsf{N}\}$,

---

[9]See also [76] for the construction of the Q-operator in the higher spin XXZ quantum chain with twisted boundary conditions.

[10]Note that there are only a few examples of integrable quantum models where the completeness has been proven in the ABA framework, including the XXX Heisenberg model; see [82] and references therein.

[11]The completeness (i.e. the non-degeneracy of spectrum) of the set of commuting conserved charges is a natural requirement to state the complete integrability of the quantum model as it represents the natural quantum analogue to the classical definition of complete integrability which requires the existence of a maximal number of independent and mutually in involution integrals of motions.

we can define the so-called Lax operator $\mathsf{L}_{0n}(\lambda) \in \mathrm{End}(\mathrm{R}_0 \otimes \mathrm{R}_n)$:

$$\mathsf{L}_{0n}(\lambda) \equiv \begin{pmatrix} A_n(\lambda) & B_n \\ C_n & D_n(\lambda) \end{pmatrix}_0 = \begin{pmatrix} x_+(\lambda) + x_-(\lambda)\sigma_n^z & (q - q^{-1})\sigma_n^-, \\ (q - q^{-1})\sigma_n^+ & x_-(\lambda) + x_+(\lambda)\sigma_n^z \end{pmatrix}_0, \tag{2.2}$$

where we have denoted $q = e^\eta \in \mathbb{C}$ and

$$x_\pm(\lambda) \equiv (\lambda q - (q\lambda)^{-1} \pm \lambda - \lambda^{-1})/2. \tag{2.3}$$

$\mathsf{L}_{0n}(\lambda)$ is a solution of the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)\mathsf{L}_{1n}(\lambda)\mathsf{L}_{2n}(\mu) = \mathsf{L}_{2n}(\mu)\mathsf{L}_{1n}(\lambda)R_{12}(\lambda/\mu), \tag{2.4}$$

w.r.t. the 6-vertex R-matrix:

$$R_{12}(\lambda) \equiv \begin{pmatrix} \lambda q - (q\lambda)^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda q - (q\lambda)^{-1} \end{pmatrix}. \tag{2.5}$$

Then, we can introduce the so-called monodromy matrix:

$$\mathsf{M}_0(\lambda) \equiv \begin{pmatrix} \mathsf{A}(\lambda) & \mathsf{B}(\lambda) \\ \mathsf{C}(\lambda) & \mathsf{D}(\lambda) \end{pmatrix} \equiv \mathsf{L}_{0\mathsf{N}}(\lambda_\mathsf{N}) \cdots \mathsf{L}_{01}(\lambda_1) \in \mathrm{End}(\mathrm{R}_0 \otimes \mathcal{R}_\mathsf{N}), \tag{2.6}$$

where $\mathcal{R}_\mathsf{N} \equiv \otimes_{i=1}^\mathsf{N} \mathrm{R}_n$, $\lambda_n \equiv \lambda/\eta_n$ and the $\eta_n \in \mathbb{C}$ are called inhomogeneity parameters. The monodromy matrix $\mathsf{M}(\lambda)$ is also a solution of the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)\mathsf{M}_1(\lambda)\mathsf{M}_2(\mu) = \mathsf{M}_2(\mu)\mathsf{M}_1(\lambda)R_{12}(\lambda/\mu), \tag{2.7}$$

and its elements A, B, C and D are the generators of a 2^N-dimensional representation of Yang-Baxter algebra on $\mathcal{R}_\mathsf{N}$.

2.1.1 Left and right representations of Yang-Baxter algebra

Let us denote with $|k, n\rangle$ the standard spin basis for the 2-dimensional linear space R_n:

$$\sigma_n^z|k, n\rangle = k|k, n\rangle, \quad k \in \{-1, 1\}, \tag{2.8}$$

i.e. the σ_n^z-eigenbasis of the local space R_n. Let L_n be the linear space dual of R_n and let $\langle k, n|$ be the elements of the dual spin basis defined by:

$$\langle k, n|k', n\rangle = (|k, n\rangle, |k', n\rangle) \equiv \delta_{k,k'} \quad \forall k, k' \in \{-1, 1\}, \tag{2.9}$$

i.e. the covectors $\langle k, n|$ define the σ_n^z-eigenbasis in the dual linear space L_n. In the *left* (covectors) and *right* (vectors) linear spaces:

$$\mathcal{L}_\mathsf{N} \equiv \otimes_{n=1}^\mathsf{N} \mathrm{L}_n, \quad \mathcal{R}_\mathsf{N} \equiv \otimes_{n=1}^\mathsf{N} \mathrm{R}_n, \tag{2.10}$$

the representations of the local $sl(2)$ generators induce left and right representations of dimension 2^N of the monodromy matrix elements, i.e. 2^N-dimensional representations of Yang-Baxter algebra with N parameters the inhomogeneities.

2.1.2 Antiperiodic transfer matrix $\bar{\mathsf{T}}(\lambda)$

The Yang-Baxter equations (2.7) and the commutation relations:

$$[R_{12}(\lambda/\mu), \Sigma_1^{(\alpha,b)} \otimes \Sigma_2^{(\alpha,b)}] = 0, \tag{2.11}$$

where:

$$\Sigma_0^{(\alpha,b)} = (\sigma_0^x)^b \begin{pmatrix} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{pmatrix}_0 \qquad \forall \alpha \in \mathbb{C}, \ b = 0, 1, \tag{2.12}$$

imply that the transfer matrix:

$$\mathsf{T}^{(\alpha,b)}(\lambda) = \text{tr}_0[\Sigma_0^{(\alpha,b)} \mathsf{M}_0(\lambda)], \tag{2.13}$$

for any fixed $\Sigma_0^{(\alpha,b)}$, generates a one-parameter family of commuting operators on $\mathcal{R}_\mathsf{N}$. Let us recall that the so-called quantum determinant:

$$\det \mathsf{M}(\lambda) \equiv \mathsf{A}(\lambda)\mathsf{D}(\lambda/q) - \mathsf{B}(\lambda)\mathsf{C}(\lambda/q), \tag{2.14}$$

is a central element[12] of the Yang-Baxter algebra (2.7) which admits the following factorized form:

$$\det \mathsf{M}(\lambda) = \prod_{n=1}^{\mathsf{N}} \det_q \mathsf{L}_{0n}(\lambda_n), \tag{2.15}$$

in terms of the local quantum determinants:

$$\det \mathsf{L}_{0n}(\lambda) \equiv A_n(\lambda)D_n(\lambda/q) - B_n C_n, \tag{2.16}$$

which explicitly reads:

$$\det \mathsf{M}(\lambda) \equiv -a(\lambda)d(\lambda/q), \quad a(\lambda) \equiv -\prod_{n=1}^{\mathsf{N}}(\frac{\lambda q}{\eta_n} - \frac{\eta_n}{\lambda q}), \quad d(\lambda) \equiv \prod_{n=1}^{\mathsf{N}}(\frac{\lambda}{\eta_n} - \frac{\eta_n}{\lambda}). \tag{2.17}$$

In the following we will analyze the spectral problem for the antiperiodic transfer matrix:

$$\bar{\mathsf{T}}(\lambda) \equiv \mathsf{B}(\lambda) + \mathsf{C}(\lambda) = \mathsf{T}^{(\alpha=0,b=1)}(\lambda), \tag{2.18}$$

then it is important to point out the conditions under which this transfer matrix is normal:

Lemma 2.1. I) *In the massless regime, i.e. for $q = e^\eta$ a pure phase ($\eta \in i\mathbb{R}$), when all the inhomogeneities $\{\eta_1, ..., \eta_\mathsf{N}\}$ are real numbers then the transfer matrix $\bar{\mathsf{T}}(\lambda)$ is a one parameter family of normal operators and the family:*

$$i\bar{\mathsf{T}}(\lambda) \tag{2.19}$$

is self-adjoint for any $\lambda q^{1/2} \in \mathbb{R}$.

II) *In the massive regime, i.e. $q = e^\eta \in \mathbb{R}^+$ ($\eta \in \mathbb{R}$), when all the inhomogeneities $\{\eta_1, ..., \eta_\mathsf{N}\}$ are pure phases then the transfer matrix $\bar{\mathsf{T}}(\lambda)$ is a one parameter family of normal operators and the family:*

$$i^{\mathsf{e}_\mathsf{N}} \bar{\mathsf{T}}(\lambda), \tag{2.20}$$

where $\mathsf{e}_\mathsf{N} = \{1$ for N even, 0 for N odd$\}$, is self-adjoint for any $\lambda q^{1/2}$ a pure phase.

---

[12]The centrality of the quantum determinant in the Yang-Baxter algebra was first discovered in [86]; see also [87] for an historical note.

Proof. In the case I) it is trivial to verify that the local Lax operators $\mathsf{L}_{0n}(\lambda)$ satisfies the following Hermitian conjugation property:

$$\mathsf{L}_{0n}(\lambda)^{\dagger} \equiv \sigma_0^y \mathsf{L}_{0n}(\lambda^*/q)\sigma_0^y, \tag{2.21}$$

where $\dagger$ means the complex conjugation plus the transposition w.r.t. the local quantum space n. Then, for the monodromy matrix it follows:

$$\mathsf{M}(\lambda)^{\dagger} \equiv \left(\begin{array}{cc} \mathsf{A}^{\dagger}(\lambda) & \mathsf{B}^{\dagger}(\lambda) \\ \mathsf{C}^{\dagger}(\lambda) & \mathsf{D}^{\dagger}(\lambda) \end{array} \right) = \left(\begin{array}{cc} \mathsf{D}(\lambda^*/q) & -\mathsf{C}(\lambda^*/q) \\ -\mathsf{B}(\lambda^*/q) & \mathsf{A}(\lambda^*/q) \end{array} \right), \tag{2.22}$$

when all the inhomogeneities $\{\eta_1, ..., \eta_N\}$ are real numbers. Then the transfer matrix $\bar{\mathsf{T}}(\lambda)$ is normal for any $\lambda \in \mathbb{C}$ and the statement in I) simply follows.

In the case II) it holds:

$$\mathsf{L}_{0n}(\lambda)^{\dagger} \equiv \sigma_0^x \mathsf{L}_{0n}(-1/(\lambda^* q))\sigma_0^x. \tag{2.23}$$

Then, for the monodromy matrix it follows:

$$\mathsf{M}(\lambda)^{\dagger} \equiv \left(\begin{array}{cc} \mathsf{A}^{\dagger}(\lambda) & \mathsf{B}^{\dagger}(\lambda) \\ \mathsf{C}^{\dagger}(\lambda) & \mathsf{D}^{\dagger}(\lambda) \end{array} \right) = \left(\begin{array}{cc} \mathsf{D}(-1/(\lambda^* q)) & \mathsf{C}(-1/(\lambda^* q)) \\ \mathsf{B}(-1/(\lambda^* q)) & \mathsf{A}(-1/(\lambda^* q)) \end{array} \right), \tag{2.24}$$

when all the inhomogeneities $\{\eta_1, ..., \eta_N\}$ are pure phases. Then the transfer matrix $\bar{\mathsf{T}}(\lambda)$ is normal for any $\lambda \in \mathbb{C}$ and the statement in II) simply follows. $\qquad \square$

2.2 Antiperiodic spin-1/2 XXZ quantum chain

In the framework of the quantum inverse scattering method, the integrability of a quantum models is proven showing that the Hamiltonian of the models belong to the one parameter families of commuting transfer matrices. In the following, we solve the spectral problem and compute matrix elements of local operators on the eigenstates of the antiperiodic transfer matrix $\bar{\mathsf{T}}(\lambda)$. It is then relevant to point out that such analysis allows in particular to describe the antiperiodic spin-1/2 XXZ quantum chain in the special case of the homogeneous limit $\eta_n \to 1$. Indeed, its Hamiltonian:

$$H = \sum_{n=1}^{N} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh\eta\, \sigma_n^z \sigma_{n+1}^z \right], \tag{2.25}$$

with the following boundary conditions:

$$\sigma_{N+1}^a = \sigma_1^x \sigma_1^a \sigma_1^x = (-1)^{1-\delta_{a,x}}\, \sigma_1^a, \quad a = x, y, z \tag{2.26}$$

where we have used the notation $\delta_{a,x} = \{1 \text{ for } a = x, 0 \text{ for } a = y, z\}$, is obtained in the homogeneous limit[13] by:

$$H = (q - q^{-1})\, \left.\frac{\partial \ln \bar{\mathsf{T}}(\lambda)}{\partial \lambda}\right|_{\lambda=1,\eta_n=1} - \mathsf{N}\frac{(q+q^{-1})}{2}. \tag{2.27}$$

---

[13]Note that substituting in (2.27) the generic transfer matrix $\mathsf{T}^{(\alpha,b)}$ we get the same Hamiltonian (2.25) where the boundary conditions are given by (2.26) with σ_1^x substituted by $\Sigma_1^{(\alpha,b)}$.

3 SOV-representations for $\bar{\mathsf{T}}(\lambda)$-spectral problem

According to Sklyanin's method [22]-[24], a separation of variable representation for the spectral problem of the transfer matrix $\bar{\mathsf{T}}(\lambda)$ is defined as a representation where the commutative family of operators $\mathsf{D}(\lambda)$ (or $\mathsf{A}(\lambda)$) is diagonal and with simple spectrum. In fact, it holds:

Theorem 3.1. *For any fixed N-tuple of inhomogeneities $\{\eta_1, ..., \eta_{\mathsf{N}}\} \in \mathbb{C}^{\mathsf{N}}$ such that*

$$\eta_a \neq q^j \eta_b \quad \forall j \in \{-1, 0, 1\}, \ a < b \in \{1, ..., \mathsf{N}\} \tag{3.1}$$

there exists a SOV representation for the $\bar{\mathsf{T}}(\lambda)$-spectral problem; i.e. $\mathsf{D}(\lambda)$ (or $\mathsf{A}(\lambda)$) is diagonalizable and with simple spectrum.

The theorem follows by the following explicit construction[14] of $\mathsf{D}(\lambda)$-eigenbasis. Let us define the left and right *references states*:

$$\langle 0| \equiv \otimes_{n=1}^{\mathsf{N}} \langle 1, n| \quad \text{and} \quad |0\rangle \equiv \otimes_{n=1}^{\mathsf{N}} |1, n\rangle, \tag{3.2}$$

then:

Theorem 3.2. I) *Left $\mathsf{D}(\lambda)$ SOV-representations:* *Under the condition* (3.1), *the states:*

$$\langle h_1, ..., h_{\mathsf{N}}| \equiv \frac{1}{\mathsf{N}} \langle 0| \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathsf{C}(\eta_n)}{d(\eta_n/q)} \right)^{h_n}, \tag{3.3}$$

where

$$\mathsf{N} = \prod_{1 \leq b < a \leq \mathsf{N}} (\eta_a/\eta_b - \eta_b/\eta_a)^{1/2} \tag{3.4}$$

$h_n \in \{0, 1\}$, $n \in \{1, ..., \mathsf{N}\}$, *define a $\mathsf{D}(\lambda)$-eigenbasis of $\mathcal{L}_{\mathsf{N}}$:*

$$\langle h_1, ..., h_{\mathsf{N}}| \mathsf{D}(\lambda) = d_{\boldsymbol{h}}(\lambda) \langle h_1, ..., h_{\mathsf{N}}|, \tag{3.5}$$

where:

$$d_{\boldsymbol{h}}(\lambda) \equiv \prod_{n=1}^{\mathsf{N}} \left(\frac{\lambda q^{h_n}}{\eta_n} - \frac{\eta_n}{\lambda q^{h_n}} \right) \quad \text{and} \quad \boldsymbol{h} \equiv (h_1, ..., h_{\mathsf{N}}). \tag{3.6}$$

The action of the remaining Yang-Baxter generators on the generic state $\langle h_1, ..., h_{\mathsf{N}}|$ reads:

$$\langle h_1, ..., h_{\mathsf{N}}| \mathsf{C}(\lambda) = \sum_{a=1}^{\mathsf{N}} \prod_{b \neq a} \frac{\lambda q^{h_b}/\eta_b - \eta_b/q^{h_b}\lambda}{\eta_a q^{(h_b - h_a)}/\eta_b - \eta_b/q^{(h_b - h_a)}\eta_a} d(\eta_a q^{h_a - 1}) \langle h_1, ..., h_{\mathsf{N}}| T_a^+, \tag{3.7}$$

$$\langle h_1, ..., h_{\mathsf{N}}| \mathsf{B}(\lambda) = \sum_{a=1}^{\mathsf{N}} \prod_{b \neq a} \frac{\lambda q^{h_b}/\eta_b - \eta_b/q^{h_b}\lambda}{\eta_a q^{(h_b - h_a)}/\eta_b - \eta_b/q^{(h_b - h_a)}\eta_a} a(\eta_a q^{h_a - 1}) \langle h_1, ..., h_{\mathsf{N}}| T_a^-, \tag{3.8}$$

where:

$$\langle h_1, ..., h_a, ..., h_{\mathsf{N}}| T_a^{\pm} = \langle h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}|. \tag{3.9}$$

Finally, $\mathsf{A}(\lambda)$ is uniquely defined by the quantum determinant relation.

---

[14] For completeness the construction of the SOV representation w.r.t. $\mathsf{A}(\lambda)$ is given in appendix.

II) *Right* $D(\lambda)$ *SOV-representations:* *Under the condition* (3.1), *the states:*

$$|h_1, ..., h_{\mathsf{N}}\rangle \equiv \frac{1}{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathsf{B}(\eta_n)}{a(\eta_n)}\right)^{h_n} |0\rangle, \tag{3.10}$$

where $h_n \in \{0, 1\}$, $n \in \{1, ..., \mathsf{N}\}$, *define a* $D(\lambda)$*-eigenbasis of* $\mathcal{R}_{\mathsf{N}}$:

$$D(\lambda)|h_1, ..., h_{\mathsf{N}}\rangle = d_{\boldsymbol{h}}(\lambda)|h_1, ..., h_{\mathsf{N}}\rangle. \tag{3.11}$$

The action of the remaining Yang-Baxter generators on the generic state $|h_1, ..., h_{\mathsf{N}}\rangle$ *reads:*

$$\mathsf{C}(\lambda)|h_1, ..., h_{\mathsf{N}}\rangle = \sum_{a=1}^{\mathsf{N}} T_a^- |h_1, ..., h_{\mathsf{N}}\rangle \prod_{b \neq a} \frac{\lambda q^{h_b}/\eta_b - \eta_b/q^{h_b}\lambda}{\eta_a q^{(h_b-h_a)}/\eta_b - \eta_b/q^{(h_b-h_a)}\eta_a} d(\eta_a q^{-h_a}), \tag{3.12}$$

$$\mathsf{B}(\lambda)|h_1, ..., h_{\mathsf{N}}\rangle = \sum_{a=1}^{\mathsf{N}} T_a^+ |h_1, ..., h_{\mathsf{N}}\rangle \prod_{b \neq a} \frac{\lambda q^{h_b}/\eta_b - \eta_b/q^{h_b}\lambda}{\eta_a q^{(h_b-h_a)}/\eta_b - \eta_b/q^{(h_b-h_a)}\eta_a} a(\eta_a q^{-h_a}). \tag{3.13}$$

where:

$$T_a^\pm |h_1, ..., h_a, ..., h_{\mathsf{N}}\rangle = |h_1, ..., h_a \pm 1, ..., h_{\mathsf{N}}\rangle. \tag{3.14}$$

Finally, $\mathsf{A}(\lambda)$ *is uniquely defined by the quantum determinant relation.*

Proof. The proof of the theorem is based on Yang-Baxter commutation relations and on the fact that the left and right references states are $D(\lambda)$-eigenstates:

$$\langle 0|\mathsf{A}(\lambda) = a(\lambda)\langle 0|, \quad \langle 0|\mathsf{D}(\lambda) = d(\lambda)\langle 0|, \quad \langle 0|\mathsf{B}(\lambda) = \underline{0}, \quad \langle 0|\mathsf{C}(\lambda) \neq \underline{0}, \tag{3.15}$$

and

$$\mathsf{A}(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad \mathsf{D}(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad \mathsf{C}(\lambda)|0\rangle = \underline{0}, \quad \mathsf{B}(\lambda)|0\rangle \neq \underline{0}. \tag{3.16}$$

Indeed, to prove that (3.3) and (3.10) are left and right eigenstates of $D(\lambda)$ with the eigenvalues (3.6), we have just to repeat the standard computations in algebraic Bethe ansatz [16]. Let us see explicitly the left case:

$$\begin{aligned}
\langle h_1, ..., h_{\mathsf{N}}|\mathsf{D}(\lambda) =& \frac{d(\lambda)}{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \left(\frac{\lambda q/\eta_n - \eta_n/q\lambda}{\lambda/\eta_n - \eta_n/\lambda}\right)^{h_n} \langle h_1, ..., h_{\mathsf{N}}| \\
&+ \sum_{n=1}^{\mathsf{N}} \left[\frac{d(\eta_n)}{\mathsf{N}} \frac{\delta_{h_n,1}(q-1/q)}{\lambda/\eta_a - \eta_a/\lambda} \prod_{a \neq n} \left(\frac{\lambda q/\eta_a - \eta_a/q\lambda}{\lambda/\eta_a - \eta_a/\lambda}\right)^{h_n} \right. \\
&\times \left. \langle 0| \prod_{a \neq n} \left(\frac{\mathsf{C}(\eta_a)}{d(\eta_a/q)}\right)^{h_a} \frac{\mathsf{C}(\lambda)}{d(\eta_n/q)} \right],
\end{aligned} \tag{3.17}$$

where we have used the Yang-Baxter commutation relation:

$$\mathsf{C}(\mu)\mathsf{D}(\lambda) = \frac{\lambda q/\mu - \mu/q\lambda}{\lambda/\mu - \mu/\lambda}\mathsf{D}(\lambda)\mathsf{C}(\mu) + \frac{q - 1/q}{\lambda/\mu - \mu/\lambda}\mathsf{D}(\mu)\mathsf{C}(\lambda). \tag{3.18}$$

In particular, the first term on the r.h.s. of (3.17) is generated by using the first term on the r.h.s. of (3.18) to commute the operator $D(\lambda)$ with all the $C(\eta_a)$ in (3.3) and finally using (3.15) to act with $D(\lambda)$ on the left reference state. The generic term n in the sum of (3.17) is obtained by using the commutativity of the $C(\mu)$ to write:

$$\langle h_1, ..., h_{\mathsf{N}}| = \frac{1}{\mathsf{N}} \langle 0| \left(\frac{\mathsf{C}(\eta_n)}{d(\eta_n/q)}\right)^{h_n} \prod_{a \neq n} \left(\frac{\mathsf{C}(\eta_a)}{d(\eta_a/q)}\right)^{h_a}, \tag{3.19}$$

9

and then commuting $D(\lambda)$ with all the $C(\eta_{a \neq n})$ by using the first term in the r.h.s. of (3.18) while for $D(\lambda)$ and $C(\eta_n)$ it is used the second term in (3.18). The result (3.5) follows being $d(\eta_n) = 0$.

Under the condition (3.1) the states $\langle h_1, ..., h_N |$ form a set of 2^N independent states and so they are a $D(\lambda)$-eigenbasis of the representation.

The action of $B(\eta_n/q^{h_n})$ and $C(\eta_n/q^{h_n})$ on the left and right states (3.3) and (3.10) follows by imposing the Yang-Baxter commutation relations and the quantum determinant relations. Then the left (3.7)-(3.8) and right (3.12)-(3.13) representations of $B(\lambda)$ and $C(\lambda)$ are just interpolation formulae which take into account that they are Laurent polynomials of degree $N - 1$, respectively even or odd for N odd or even. $\qquad \square$

Remark 1. It is worth remarking that representations of the type (3.7)-(3.8) and (3.12)-(3.13) for the generators of the 6-vertex Yang-Baxter algebra can be also derived from the original representations by implementing the change of basis prescribed from the factorizing F-matrices [88]. Let us recall that these F-matrices were introduced to provide explicit representations of the Drinfel'd's twist of quasi-triangular quasi-Hopf algebras [89]-[91]. The connection with Sklyanin's quantum separation of variables in its functional version was recognized in [92] and there used to construct the factorizing F-matrices for general Yangian $Y(sl(2))$; i.e. the rational 6-vertex Yang-Baxter algebra associated to the general spin s quantum chain representations. These results were used by the Lyon group in [30] mainly as tools to get the solution of the quantum inverse problem and to re-derive the Slavnov's scalar product formula [93, 94] for the periodic spin-1/2 XXZ quantum chain but not to solve the corresponding spectral problem. This is natural as these representations do not define quantum separate variables representations for the spectral problem associated to the transfer matrix of the periodic chain.

3.1 Sklyanin's measure and scalar products

In the next subsection, we compute the coupling between states belonging to right and left SOV-basis. We show that up to an overall constant these are completely fixed by the left and right SOV-representations of the Yang-Baxter algebras when the gauge in the SOV-representations are chosen. Then, we use these results to compute the scalar products between states which in the left and right SOV-basis have a *separated form* similar to that of the transfer matrix eigenstates. The resulting scalar product formula admits a determinant representation which can be considered as the SOV analogous of the Slavnov's scalar product formula computed for Bethe states in the framework of the algebraic Bethe ansatz.

3.1.1 Coupling of left and right SOV-basis

It may be helpful to present the main properties of the matrices which define the change of basis from the original basis to the SOV-basis; this will lead us to introduce naturally the coupling between pairs of states belonging to left and right SOV-basis and the concept of *Sklyanin's measure* in our model. Let us define the following isomorphism:

$$\varkappa : (h_1, ..., h_N) \in \{0, 1\}^N \rightarrow j = \varkappa(h_1, ..., h_N) \equiv 1 + \sum_{a=1}^{N} 2^{(a-1)} h_a \in \{1, ..., 2^N\}, \qquad (3.20)$$

then we can write:

$$\langle \mathbf{y}_j | = \langle \mathbf{x}_j | U^{(L)} = \sum_{i=1}^{2^N} U_{j,i}^{(L)} \langle \mathbf{x}_i | \quad \text{and} \quad | \mathbf{y}_j \rangle = U^{(R)} | \mathbf{x}_j \rangle = \sum_{i=1}^{2^N} U_{i,j}^{(R)} | \mathbf{x}_i \rangle, \qquad (3.21)$$

where we have used the notations:

$$\langle \mathbf{y}_j | \equiv \langle h_1, ..., h_\mathsf{N}| \text{ and } |\mathbf{y}_j\rangle \equiv |h_1, ..., h_\mathsf{N}\rangle, \tag{3.22}$$

to represent, respectively, the states of the left and right SOV-basis and:

$$\langle \mathbf{x}_j | \equiv \otimes_{n=1}^{\mathsf{N}} \langle 2h_n - 1, n| \quad \text{and} \quad |\mathbf{x}_j\rangle \equiv \otimes_{n=1}^{\mathsf{N}} |2h_n - 1, n\rangle, \tag{3.23}$$

to represent, respectively, the states of the left and right original σ_n^z-orthonormal basis. Here, $U^{(L)}$ and $U^{(R)}$ are the $2^\mathsf{N} \times 2^\mathsf{N}$ matrices for which it holds:

$$U^{(L)}\mathsf{D}(\lambda) = \Delta_\mathsf{D}(\lambda)U^{(L)}, \quad \mathsf{D}(\lambda)U^{(R)} = U^{(R)}\Delta_\mathsf{D}(\lambda), \tag{3.24}$$

where $\Delta_\mathsf{D}(\lambda)$ is a diagonal $2^\mathsf{N} \times 2^\mathsf{N}$ matrix. The diagonalizability and simplicity of the D-spectrum imply the invertibility of the matrices $U^{(L)}$ and $U^{(R)}$ and the fact that all the diagonal entry of $\Delta_\mathsf{D}(\lambda)$ are Laurent polynomials in λ with different zeros. Then the following proposition holds:

Proposition 3.1. *The $2^\mathsf{N} \times 2^\mathsf{N}$ matrix:*

$$M \equiv U^{(L)}U^{(R)} \tag{3.25}$$

is diagonal and it is characterized by:

$$M_{jj} = \langle \mathbf{y}_j | \mathbf{y}_j \rangle = \langle h_1, ..., h_\mathsf{N} | h_1, ..., h_\mathsf{N} \rangle = \prod_{1 \le b < a \le \mathsf{N}} \frac{1}{\eta_a q^{(h_b - h_a)}/\eta_b - \eta_b/q^{(h_b - h_a)}\eta_a}. \tag{3.26}$$

Proof. The fact that the matrix M is diagonal is a trivial consequence of the *orthogonality* of left and right eigenstates corresponding to different eigenvalue of $\mathsf{D}(\lambda)$.

Let us compute the matrix element $\theta_a \equiv \langle h_1, ..., h_a = 0, ..., h_\mathsf{N} | \mathsf{C}(\eta_a) | h_1, ..., h_a = 1, ..., h_\mathsf{N} \rangle$, where $a \in \{1, ..., \mathsf{N}\}$. Then using the left action of the operator $\mathsf{C}(\eta_a)$ we get:

$$\theta_a = d(\eta_a/q)\langle h_1, ..., h_a = 1, ..., h_\mathsf{N} | h_1, ..., h_a = 1, ..., h_\mathsf{N} \rangle, \tag{3.27}$$

while using the right action of the operator $\mathsf{C}(\eta_a)$ and the orthogonality of right and left D-eigenstates corresponding to different eigenvalues we get:

$$\theta_a = \prod_{b \ne a, b=1}^{\mathsf{N}} \frac{(\eta_a q^{h_b}/\eta_b - \eta_b/q^{h_b}\eta_a)}{(\eta_a q^{h_b - 1}/\eta_b - \eta_b/q^{h_b - 1}\eta_a)} d(\eta_a/q)\langle h_1, ..., h_a = 0, ..., h_\mathsf{N} | h_1, ..., h_a = 0, ..., h_\mathsf{N} \rangle \tag{3.28}$$

and so:

$$\frac{\langle h_1, ..., h_a = 1, ..., h_\mathsf{N} | h_1, ..., h_a = 1, ..., h_\mathsf{N} \rangle}{\langle h_1, ..., h_a = 0, ..., h_\mathsf{N} | h_1, ..., h_a = 0, ..., h_\mathsf{N} \rangle} = \prod_{b \ne a, b=1}^{\mathsf{N}} \frac{(\eta_a q^{h_b}/\eta_b - \eta_b/q^{h_b}\eta_a)}{(\eta_a q^{h_b - 1}/\eta_b - \eta_b/q^{h_b - 1}\eta_a)}. \tag{3.29}$$

The previous formula implies:

$$\frac{\langle h_1, ..., h_\mathsf{N} | h_1, ..., h_\mathsf{N} \rangle}{\langle 0|0\rangle/\mathsf{N}^2} = \prod_{1 \le b < a \le \mathsf{N}} \frac{\eta_a/\eta_b - \eta_b/\eta_a}{\eta_a q^{(h_b - h_a)}/\eta_b - \eta_b/q^{(h_b - h_a)}\eta_a}, \tag{3.30}$$

from which the proposition follows recalling the definition (3.4) of the normalization N and remarking that in our choice of the original spin basis it holds:

$$\langle 0|0\rangle = 1. \tag{3.31}$$

$\square$

11

3.1.2 SOV-decomposition of the identity

The previous results allow to write the following spectral decomposition of the identity $\mathbb{I} \in \mathrm{End}(\mathcal{R}_\mathsf{N})$:

$$\mathbb{I} \equiv \sum_{i=1}^{2^\mathsf{N}} \mu_i |\mathbf{y}_i\rangle\langle\mathbf{y}_i|, \tag{3.32}$$

in terms of the left and right SOV-basis. Here,

$$\mu_i \equiv \frac{1}{\langle\mathbf{y}_i|\mathbf{y}_i\rangle}, \tag{3.33}$$

is the so-called *Sklyanin's measure*; which is a discrete measure in the XXZ spin 1/2 quantum chain. Explicitly, the SOV-decomposition of the identity reads:

$$\mathbb{I} \equiv \sum_{h_1,\ldots,h_\mathsf{N}=0}^{1} \prod_{1 \le b < a \le \mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b}) \frac{|h_1,\ldots,h_\mathsf{N}\rangle\langle h_1,\ldots,h_\mathsf{N}|}{\prod_{b=1}^{\mathsf{N}} \omega(\eta_b q^{h_b})}, \tag{3.34}$$

where:

$$\omega(\eta) \equiv \eta^{\mathsf{N}-1}, \tag{3.35}$$

and they are gauge dependent parameters.

Remark 2. Sklyanin's measure[15] has been first introduced by Sklyanin in his article on quantum Toda chain [22]. There it has been derived as a consequence of the self-adjointness of the transfer matrix w.r.t. the scalar product. In particular, the Hermitian properties of the operator zeros and their conjugate shift operators have been fixed to assure the self-adjointness of the transfer matrix. In the similar but more involved non-compact case of the Sinh-Gordon model [96], the problem related to the uniqueness of the definition of this measure has been analyzed. There it has been proven that the measure is in fact uniquely determined once the positive self-adjointness of the generators $\mathsf{A}(\lambda)$ and $\mathsf{D}(\lambda)$ is required. The approach here used is suitable for general compact SOV-representations of 6-vertex Yang-Baxter algebra.

3.2 SOV characterization of $\bar{\mathsf{T}}(\lambda)$-spectrum

Let us denote with $\Sigma_{\bar{\mathsf{T}}}$ the set of the eigenvalue functions $t(\lambda)$ of the transfer matrix $\bar{\mathsf{T}}(\lambda)$, then $\Sigma_{\bar{\mathsf{T}}}$ is contained in:

$$\mathbb{C}_{even}[\lambda, \lambda^{-1}]_{\mathsf{N}-1} \text{ for } \mathsf{N} \text{ odd}, \quad \mathbb{C}_{odd}[\lambda, \lambda^{-1}]_{\mathsf{N}-1} \text{ for } \mathsf{N} \text{ even}, \tag{3.36}$$

where $\mathbb{C}_\epsilon[x, x^{-1}]_\mathsf{M}$ denotes the linear space in the field $\mathbb{C}$ of the Laurent polynomials of degree M in the variable x which are even or odd as stated in the index ϵ.

Theorem 3.3. *If the inhomogeneities parameters* $\{\eta_1, \ldots, \eta_\mathsf{N}\}$ *satisfy the conditions* (3.1)*, then the spectrum of* $\bar{\mathsf{T}}(\lambda)$ *is simple and* $\Sigma_{\bar{\mathsf{T}}}$ *coincides with the set of functions in* (3.36) *which are solutions of the discrete system of equations:*

$$t(\eta_a)t(\eta_a/q) = a(\eta_a)d(\eta_a/q), \quad \forall a \in \{1, \ldots, \mathsf{N}\}. \tag{3.37}$$

---

[15]See also [95] for further discussions on the measure.

I) The right $\bar{\mathsf{T}}$-eigenstate corresponding to a $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ is characterized by:

$$|t\rangle = \sum_{h_1,...,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \frac{Q_t(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})} \prod_{1 \le b < a \le \mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b})|h_1, ..., h_{\mathsf{N}}\rangle, \tag{3.38}$$

where, up to an overall normalization of the state, the coefficients are characterized by:

$$Q_t(\eta_a/q)/Q_t(\eta_a) = t(\eta_a)/d(\eta_a/q). \tag{3.39}$$

II) The left $\bar{\mathsf{T}}$-eigenstate corresponding to $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ is characterized by:

$$\langle t| = \sum_{h_1,...,h_{\mathsf{N}}=0}^{1} \prod_{a=1}^{\mathsf{N}} \frac{\bar{Q}_t(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})} \prod_{1 \le b < a \le \mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b})\langle h_1, ..., h_{\mathsf{N}}|, \tag{3.40}$$

where up to an overall normalization of the state the coefficients are characterized by:

$$\bar{Q}_t(\eta_a/q)/\bar{Q}_t(\eta_a) = t(\eta_a)/a(\eta_a). \tag{3.41}$$

Proof. In the SOV representations the spectral problem for $\bar{\mathsf{T}}(\lambda)$ is reduced to a discrete system of 2^{N} Baxter-like equations. Indeed, let $\langle t|$ be a $\bar{\mathsf{T}}$-eigenstate corresponding to the eigenvalue $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$, then the coefficients (*wave-functions*)

$$\Psi_t(\mathbf{h}) \equiv \langle t| h_1, ..., h_{\mathsf{N}}\rangle \tag{3.42}$$

of $\langle t|$ in the SOV-basis satisfy the equations:

$$t(\eta_n q^{-h_n})\Psi_t(\mathbf{h}) = a(\eta_n q^{-h_n})\Psi_t(\mathsf{T}_n^+(\mathbf{h})) + d(\eta_n q^{-h_n})\Psi_t(\mathsf{T}_n^-(\mathbf{h})), \tag{3.43}$$

for any $n \in \{1, ..., \mathsf{N}\}$ and $\mathbf{h} \in \{0, 1\}^{\mathsf{N}}$, where we have denoted:

$$\mathsf{T}_n^{\pm}(\mathbf{h}) \equiv (h_1, \ldots, h_n \pm 1, \ldots, h_{\mathsf{N}}). \tag{3.44}$$

Taking into account that:

$$a(\eta_n/q) = d(\eta_n) = 0, \tag{3.45}$$

the previous system of equations can be rewritten as a system of homogeneous equations:

$$\begin{pmatrix} t(\eta_n) & -a(\eta_n) \\ -d(\eta_n/q) & t(\eta_n/q) \end{pmatrix} \begin{pmatrix} \Psi_t(h_1, ..., h_n = 0, ..., h_1) \\ \Psi_t(h_1, ..., h_n = 1, ..., h_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.46}$$

for any $n \in \{1, ..., \mathsf{N}\}$ with $h_{m \ne n} \in \{0, 1\}$. Note that the condition $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ implies that the previous system has to have a non-trivial solution, i.e. the determinants of the 2×2 matrices in (3.46) must be zero for any $n \in \{1, ..., \mathsf{N}\}$. So that the condition $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ implies (3.37). Now let us observe that being

$$a(\eta_n) \ne 0 \text{ and } d(\eta_n/q) \ne 0, \tag{3.47}$$

the rank of the matrices in (3.46) is 1 and then up to an overall normalization the solution is unique:

$$\frac{\Psi_t(h_1, ..., h_n = 1, ..., h_1)}{\Psi_t(h_1, ..., h_n = 0, ..., h_1)} = \frac{t(\eta_a)}{a(\eta_a)}, \tag{3.48}$$

13

for any $n \in \{1, ..., \mathsf{N}\}$ with $h_{m \neq n} \in \{0, 1\}$. This implies that given a $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ there exist (up to normalization) one and only one corresponding $\bar{\mathsf{T}}$-eigenstate $\langle t|$ with coefficients which have the factorized form given in (3.40)-(3.41) and then the $\bar{\mathsf{T}}$-spectrum is simple.

Vice versa, let us take a $t(\lambda)$ in the set of functions (3.36) which is solution of the system (3.37) then for the state $\langle t|$ constructed by (3.40)-(3.41) it holds:

$$\langle t| \, \bar{\mathsf{T}}(\eta_n q^{-h_n})|h_1, ..., h_{\mathsf{N}}\rangle = t(\eta_n q^{-h_n}) \, \langle t| \, h_1, ..., h_{\mathsf{N}}\rangle \;\; \forall n \in \{1, ..., \mathsf{N}\}, \tag{3.49}$$

for any D-eigenstate $|h_1, ..., h_{\mathsf{N}}\rangle$. Then being $\bar{\mathsf{T}}(\lambda)$ a Laurent polynomials of degree $\mathsf{N} - 1$ in λ, respectively even or odd for N odd or even, it follows:

$$\langle t| \, \bar{\mathsf{T}}(\lambda)|h_1, ..., h_{\mathsf{N}}\rangle = t(\lambda) \, \langle t| \, h_1, ..., h_{\mathsf{N}}\rangle, \tag{3.50}$$

that is $t(\lambda) \in \Sigma_{\bar{\mathsf{T}}}$ and $\langle t|$ is the corresponding $\bar{\mathsf{T}}$-eigenstate. $\qquad\square$

The previous theorem gives a well defined and complete characterization of the spectrum of the transfer matrix $\bar{\mathsf{T}}(\lambda)$ and the normality of $\bar{\mathsf{T}}(\lambda)$ implies that the discrete system of equations has to admit 2^{N} independent solutions in the class of functions (3.36). However, it is worth pointing out that such a characterization of the spectrum is not the most efficient; in particular, for the analysis of the continuum limit. A reformulation of the SOV characterization of the $\bar{\mathsf{T}}$-spectrum by functional equations is then important and it can be achieved by the construction of a Baxter Q-operator whose functional equation, computed in the spectrum of the D-zeros, coincides with the finite system of Baxter-like equations (3.43). For the model under consideration a Q-operator has been constructed in [77] and it satisfies[16]:

$$\left[\bar{\mathsf{T}}(\lambda), Q(\lambda)\right] = 0, \;\; [Q(\lambda), Q(\mu)] = 0, \;\; \bar{\mathsf{T}}(\lambda)Q(\lambda) = a\,(\lambda)\,Q(\lambda/q) + d\,(\lambda)\,Q(\lambda q). \tag{3.51}$$

Moreover, $Q(\lambda)$ is a Laurent polynomial with eigenvalues of the form:

$$\mathsf{Q}(\lambda) = \lambda^{-\mathsf{N}/2} \prod_{k=1}^{\mathsf{N}} (\lambda - \lambda_k), \tag{3.52}$$

where the $\{\lambda_j\}$ are solutions of the Bethe equations:

$$\prod_{n=1}^{\mathsf{N}} \frac{q^2 \lambda_k^2 - \eta_n^2}{\lambda_k^2 - \eta_n^2} = -\prod_{a=1}^{\mathsf{N}} \frac{q\lambda_k - \lambda_a}{\lambda_k/q - \lambda_a} \;\; \forall k \in \{1, ..., \mathsf{N}\}, \tag{3.53}$$

as a consequence of the requirement of analyticity of the transfer matrix eigenvalues.

Let us observe that for $q = e^{2i\pi p'/p}$ ($p, p' \in Z^{\geq 0}$) a p-root of unit the consistence condition (i.e. the existence of non-trivial solutions) for the Baxter eigenvalue functional equation lead to the functional equation[17]:

$$\det_p D(\Lambda) = 0, \;\; \Lambda \in \mathbb{C} \tag{3.54}$$

---

[16]If we translate it in our notations and we introduce the inhomogeneities.

[17]This characterization open the possibility to construct solutions of the Baxter equation in terms of cofactors of the matrix $D(\lambda)$ as explained in [34].

involving only the $\bar{\mathsf{T}}$-eigenvalue $t(\lambda)$, where $\Lambda = \lambda^p$ and $D(\lambda)$ is the $p \times p$ matrix:

$$
D(\lambda) \equiv \begin{pmatrix}
t(\lambda) & -d(\lambda) & 0 & \cdots & 0 & -a(\lambda) \\
-a(q\lambda) & t(q\lambda) & -d(q\lambda) & 0 & \cdots & 0 \\
0 & \ddots & & & & \vdots \\
\vdots & & \cdots & & & \vdots \\
\vdots & & & \cdots & & \vdots \\
\vdots & & & & \ddots & 0 \\
0 & \cdots & 0 & -a(q^{p-2}\lambda) & t(q^{p-2}\lambda) & -d(q^{p-2}\lambda) \\
-d(q^{p-1}\lambda) & 0 & \cdots & 0 & -a(q^{p-1}\lambda) & t(q^{p-1}\lambda)
\end{pmatrix}. \tag{3.55}
$$

It is worth remarking that the equation (3.54) coincides with the equation obtained by combining the fusion of transfer matrices [97, 98] and the truncation identity which holds at the root of unit[18].

4 Scalar products

Let $\langle\alpha|$ and $|\beta\rangle$ be two arbitrary left and right *separate states* which by definition have the following factorized form in the SOV-representation:

$$
\langle\alpha| = \sum_{h_1,\ldots,h_\mathsf{N}=0}^{1} \prod_{a=1}^{\mathsf{N}} \frac{\alpha_a(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})} \prod_{1 \le b < a \le \mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b}) \langle h_1,\ldots,h_\mathsf{N}|, \tag{4.1}
$$

$$
|\beta\rangle = \sum_{h_1,\ldots,h_\mathsf{N}=0}^{1} \prod_{a=1}^{\mathsf{N}} \frac{\beta_a(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})} \prod_{1 \le b < a \le \mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b}) |h_1,\ldots,h_\mathsf{N}\rangle, \tag{4.2}
$$

The interest toward these kind of states is due to the following:

Proposition 4.1. *The two states[19] $(\langle\alpha|)^\dagger$ and $|\beta\rangle$ have the following scalar product:*

$$
\langle\alpha|\beta\rangle = \det_{\mathsf{N}} ||\mathcal{M}_{a,b}^{(\alpha,\beta)}|| \quad \text{with} \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv (\eta_a)^{2(b-1)} \sum_{h=0}^{1} \frac{\alpha_a(\eta_a q^{-h}) \beta_a(\eta_a q^{-h})}{\omega(\eta_a q^{-h})} q^{-2(b-1)h}. \tag{4.3}
$$

Proof. From the SOV-decomposition, we have:

$$
\langle\alpha|\beta\rangle = \sum_{h_1,\ldots,h_\mathsf{N}=0}^{1} V\left(\left(\eta_1/q^{h_1}\right)^2,\ldots,\left(\eta_\mathsf{N}/q^{h_\mathsf{N}}\right)^2\right) \prod_{a=1}^{\mathsf{N}} \frac{\alpha_a(\eta_a q^{-h_a}) \beta_a(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})}, \tag{4.4}
$$

where $V(x_1,\ldots,x_\mathsf{N}) \equiv \prod_{1 \le b < a \le \mathsf{N}}(x_a - x_b)$ is the Vandermonde determinant. From this formula by using the multilinearity of the determinant w.r.t. the rows we prove the proposition. $\qquad\square$

Note that the form of determinant for these scalar products is not restricted to the case in which one of the two states is an eigenstate of the transfer matrix; on the contrary to what happens for the scalar product formulae in the framework of the algebraic Bethe ansatz. It is worth noticing that from the scalar product formula we can prove:

---

[18] This method was first developed for the RSOS model in [99] and was adapted to spin chains in [100, 101].

[19] Note that by using the Hermitian conjugation properties of the Yang-Baxter generators the vector $(\langle\alpha|)^\dagger \in \mathcal{R}_\mathsf{N}$ can simply be written in the $\mathsf{A}(\lambda)$ SOV-basis, see appendix for the definition of this basis. The formula (4.3) also describes the action of the covector $\langle\alpha|$ on the vector $|\beta\rangle$.

Corollary 4.1. *Transfer matrix eigenstates corresponding to different eigenvalues are orthogonal states.*

Proof. Let us denote with $|t\rangle$ and $|t'\rangle$ two eigenstates of $\bar{\mathsf{T}}(\lambda)$ with eigenvalues $t(\lambda)$ and $t'(\lambda)$. To prove the corollary, we have to prove that:

$$\det_{\mathsf{N}} ||\Phi_{a,b}^{(t,t')}|| = 0 \quad \text{with} \quad \Phi_{a,b}^{(t,t')} \equiv (\eta_a)^{2(b-1)} \sum_{c=0}^{1} \frac{Q_{t'}(\eta_a q^{-c})\bar{Q}_t(\eta_a q^{-c})}{\omega(\eta_a q^{-c})} q^{-2(b-1)c}. \tag{4.5}$$

It is enough to show the existence of a non-zero vector $\mathsf{V}^{(t,t')}$ such that:

$$\sum_{b=1}^{\mathsf{N}} \Phi_{a,b}^{(t,t')} \mathsf{V}_b^{(t,t')} = 0 \quad \forall a \in \{1,...,\mathsf{N}\}. \tag{4.6}$$

The transfer matrix eigenvalues are Laurent polynomials of degree N-1 (even for N-1 even and odd for N-1 odd) of the form:

$$t(\lambda) = \sum_{b=1}^{\mathsf{N}} c_b \lambda^{-\mathsf{N}-1+2b}, \quad t'(\lambda) = \sum_{b=1}^{\mathsf{N}} c_b' \lambda^{-\mathsf{N}-1+2b}, \tag{4.7}$$

so if we define:

$$\mathsf{V}_b^{(t,t')} \equiv c_b' - c_b \quad \forall b \in \{1,...,\mathsf{N}\}, \tag{4.8}$$

it results:

$$\sum_{b=1}^{\mathsf{N}} \Phi_{a,b}^{(t,t')} \mathsf{V}_b^{(t,t')} = \sum_{c=0}^{1} Q_{t'}(\eta_a q^{-c})\bar{Q}_t(\eta_a q^{-c})(t'(\eta_a q^{-c}) - t(\eta_a q^{-c})). \tag{4.9}$$

We can use now the discrete system of Baxter equations satisfied by the $Q_{t'}(\eta_a q^{-h_a})$ and $\bar{Q}_t(\eta_a q^{-h_a})$ to rewrite:

$$Q_{t'}(\eta_a)\bar{Q}_t(\eta_a)(t'(\eta_a) - t(\eta_a)) = a(\eta_a)Q_{t'}(\eta_a/q)\bar{Q}_t(\eta_a) - d(\eta_a/q)Q_{t'}(\eta_a)\bar{Q}_t(\eta_a/q), \tag{4.10}$$

and

$$Q_{t'}(\eta_a/q)\bar{Q}_t(\eta_a/q)(t'(\eta_a/q) - t(\eta_a/q)) = d(\eta_a/q)Q_{t'}(\eta_a)\bar{Q}_t(\eta_a/q) - a(\eta_a)Q_{t'}(\eta_a/q)\bar{Q}_t(\eta_a) \tag{4.11}$$

and by substituting them in (4.9) we get (4.6). $\qquad\square$

5 Reconstruction of local operators

The first reconstruction of local operators has been achieved in [30], for the case of the XXZ spin 1/2 chain. In [31], then the solution has been extended to fundamental lattice models, i.e. those with isomorphic auxiliary and local quantum space, for which the monodromy matrix becomes the permutation operator at a special value of the spectral parameter. Here, we present a simple modification of the reconstruction formula of [30] to adapt it to the current antiperiodic case.

Proposition 5.1. *The following reconstruction holds in terms of the antiperiodic transfer matrix:*

$$X_n = \prod_{b=1}^{n-1} \bar{\mathsf{T}}(\eta_b) tr_0(\mathsf{M}_0(\eta_n)X_0\sigma_0^x) \prod_{b=1}^{n} \frac{\bar{\mathsf{T}}(\eta_b/q)}{\det \bar{\mathsf{M}}(\eta_b)} \tag{5.1}$$

$$= \prod_{b=1}^{n} \bar{\mathsf{T}}(\eta_b) \frac{tr_0(\sigma_0^{(z)}\mathsf{M}_0^{t_0}(\eta_n/q)\sigma_0^z X_0\sigma_0^x)}{\det \mathsf{M}(\eta_n)} \prod_{b=1}^{n-1} \frac{\bar{\mathsf{T}}(\eta_b/q)}{\det \bar{\mathsf{M}}(\eta_b)}, \tag{5.2}$$

where it holds:

$$\bar{\mathsf{T}}(\eta_n)\bar{\mathsf{T}}(\eta_n/q) = \det \bar{\mathsf{M}}(\eta_b) = \mathsf{B}(\eta_b)\mathsf{C}(\eta_b/q) - \mathsf{A}(\eta_b)\mathsf{D}(\eta_b/q). \tag{5.3}$$

Proof. Here, we present a proof based on the following known [30] reconstruction in terms of the periodic transfer matrix:

$$X_n = \prod_{b=1}^{n-1} \mathsf{T}(\eta_b) \text{tr}_0(\mathsf{M}_0(\eta_n) X_0) \prod_{b=1}^{n} \frac{\mathsf{T}(\eta_b/q)}{\det \mathsf{M}(\eta_b)} \tag{5.4}$$

$$= \prod_{b=1}^{n} \mathsf{T}(\eta_b) \frac{\text{tr}_0(\sigma_0^y \mathsf{M}_0^{t_0}(\eta_n/q) \sigma_0^y X_0)}{\det \mathsf{M}(\eta_n)} \prod_{b=1}^{n-1} \frac{\mathsf{T}(\eta_b/q)}{\det \mathsf{M}(\eta_b)}, \tag{5.5}$$

where it holds:

$$\mathsf{T}(\eta_b)\mathsf{T}(\eta_b/q) = \det \mathsf{M}(\eta_b) = \mathsf{A}(\eta_b)\mathsf{D}(\eta_b/q) - \mathsf{B}(\eta_b)\mathsf{C}(\eta_b/q). \tag{5.6}$$

From the above formulae it holds:

$$\sigma_n^x \overset{(5.4)}{=} \prod_{b=1}^{n-1} \mathsf{T}(\eta_b)\bar{\mathsf{T}}(\eta_n) \prod_{b=1}^{n} \frac{\mathsf{T}(\eta_b/q)}{\det \mathsf{M}(\eta_b)} \overset{(5.5)}{=} -\prod_{b=1}^{n} \frac{\mathsf{T}(\eta_b)}{\det \mathsf{M}(\eta_b)}\bar{\mathsf{T}}(\eta_n/q) \prod_{b=1}^{n-1} \mathsf{T}(\eta_b/q), \tag{5.7}$$

from which:

$$1 = \sigma_n^x \sigma_n^x = -\prod_{b=1}^{n-1} \mathsf{T}(\eta_b) \frac{\bar{\mathsf{T}}(\eta_n)\bar{\mathsf{T}}(\eta_n/q)}{\det \mathsf{M}(\eta_n)} \prod_{b=1}^{n-1} \frac{\mathsf{T}(\eta_b/q)}{\det \mathsf{M}(\eta_b)}, \tag{5.8}$$

which implies (5.3) thanks to (5.6). Moreover, we can use (5.7) to write:

$$\prod_{b=1}^{c-1} \sigma_b^x = \prod_{b=1}^{c-1} \bar{\mathsf{T}}(\eta_b) \prod_{b=1}^{c-1} \frac{\mathsf{T}(\eta_b/q)}{\det \mathsf{M}(\eta_b)} = \prod_{b=1}^{c-1} \frac{\mathsf{T}(\eta_b)}{\det \bar{\mathsf{M}}(\eta_b)} \prod_{b=1}^{c-1} \bar{\mathsf{T}}(\eta_n/q), \tag{5.9}$$

then the result (5.1) follows by computing:

$$X_n = \prod_{b=1}^{n-1} \sigma_b^x \tilde{X}_n \prod_{b=1}^{n} \sigma_b^x \quad \text{with} \quad \tilde{X}_n = X_n \sigma_n^x \tag{5.10}$$

by using for $\tilde{X}_n$ the reconstruction (5.4) and for the first product of σ_b^x the first reconstruction in (5.9) while for the second product of σ_b^x the second reconstruction in (5.9). Similarly, the result (5.2) follows by computing:

$$X_n = \prod_{b=1}^{n} \sigma_b^x \bar{X}_n \prod_{b=1}^{n-1} \sigma_b^x \quad \text{with} \quad \bar{X}_n = \sigma_n^x X_n, \tag{5.11}$$

by using for $\bar{X}_n$ the reconstruction (5.5). $\qquad\square$

6 Form factors of the local operators

6.1 Preliminary comments

In the following we will compute the matrix elements (form factors):

$$\langle t|O_n|t'\rangle \tag{6.1}$$

which by definition are the action of the transfer matrix eigencovector $\langle t| \in \mathcal{L}_\mathsf{N}$ on the vector obtained by the action of some local spin operator O_n on the transfer matrix eigenvector $|t'\rangle \in \mathcal{R}_\mathsf{N}$, where $|t'\rangle$ and $\langle t|$ are defined in

(3.38) and (3.40). As explicitly stated in Theorem 3.3, these states are by definition characterized up to an overall normalization, then it is worth pointing out that these normalizations do not lead to limitations in the use of the form factors (6.1) to expand m-point functions like:

$$\frac{\langle t|O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle}{\langle t|t\rangle},\qquad(6.2)$$

where we are denoting with $\langle t|t\rangle$ the action of the covector $\langle t|$ on the vector $|t\rangle$ as defined in (4.3) and as well as with $\langle t|O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle$ the action of the covector $\langle t|$ on the vector $O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle$. Indeed, it is enough to remark that the m-point functions of the type (6.2) are normalization independent and moreover the following formula:

$$\mathbb{I} = \sum_{t(\lambda)\in\sum_{\mathsf{T}}} \frac{|t\rangle\langle t|}{\langle t|t\rangle},\qquad(6.3)$$

is a well defined decomposition of the identity from the diagonalizability and simplicity of the transfer matrix spectrum. Then, we can write:

$$\frac{\langle t|O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle}{\langle t|t\rangle} = \sum_{t_1(\lambda),\dots,t_{\mathrm{m}-1}(\lambda)\in\sum_{\mathsf{T}}} \frac{\langle t|O_{n_1}|t_1\rangle\langle t_{\mathrm{m}-1}|O_{n_{\mathrm{m}}}|t\rangle\prod_{a=2}^{\mathrm{m}-1}\langle t_{a-1}|O_{n_a}|t_a\rangle}{\langle t|t\rangle\prod_{a=1}^{\mathrm{m}-1}\langle t_a|t_a\rangle},\qquad(6.4)$$

where in the r.h.s there are the matrix elements that we compute in this paper.

In the representations which defines a normal transfer matrix $\bar{\mathsf{T}}(\lambda)$ it is worth remarking that the issue of the relative normalization between eigencovector and eigenvector of $\bar{\mathsf{T}}(\lambda)$ becomes more important. Indeed, taken the generic eigenvector $|t\rangle$ then the covector $\underline{\langle t|} \equiv (|t\rangle)^{\dagger}$, dual to $|t\rangle$ w.r.t. the Hermitian conjugation †, is itself an eigencovector of $\bar{\mathsf{T}}(\lambda)$ which for the simplicity of $\bar{\mathsf{T}}$-spectrum implies the following identity $\underline{\langle t|}\equiv\alpha_t\langle t|$, where $\langle t|$ is the eigencovector defined in (3.40). Of course, in these representations the following identities hold:

$$\frac{\langle t|O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle}{\langle t|t\rangle} = \frac{\underline{\langle t|}O_{n_1}\cdots O_{n_{\mathrm{m}}}|t\rangle}{\||t\rangle\|^2}\qquad(6.5)$$

where $\||t\rangle\|$ is the positive norm of the eigenvector $|t\rangle$ in the Hilbert space $\mathcal{R}_{\mathsf{N}}$ w.r.t. the scalar product introduced in Section 2.1.1. Then the form factor expansion defined in (6.4) can be used as well to compute the m-point functions for the standard definition in the Hilbert space $\mathcal{R}_{\mathsf{N}}$.

Let us finally comment that from the above discussion emerges clearly the relevance to compute explicitly the norm of the transfer matrix eigenvectors $|t\rangle$, defined in (3.38), as it allows to fix the relative normalization α_t of left and right transfer matrix eigenstates thanks to the identity $\alpha_t = \||t\rangle\|^2/\langle t|t\rangle$ in this way making possible to take these left and right states as one the exact dual of the other; this interesting issue is currently under analysis.

6.2 Results

Here, we present the main results of the present paper:

Theorem 6.1. *Let $\langle t|$ and $|t'\rangle$ be a left and a right eigenstate of the transfer matrix $\bar{\mathsf{T}}(\lambda)$, respectively, then it holds:*

$$\langle t|\sigma_n^-|t'\rangle = \frac{\prod_{h=1}^{n-1}t(\eta_h)\prod_{h=1}^{n}t'(\eta_h/q)}{\prod_{h=1}^{n}a(\eta_h)d(\eta_h/q)}\det_{\mathsf{N}+1}(\|\mathcal{S}_{a,b}^{(-,t,t')}\|)\qquad(6.6)$$

where $\|\mathcal{S}_{a,b}^{(-,t,t')}\|$ is the $(\mathsf{N}+1)\times(\mathsf{N}+1)$ matrix:

$$\mathcal{S}_{a,b}^{(-,t,t')} \equiv \Phi_{a,b+1/2}^{(t,t')} \;\; for \; a \in \{1,...,\mathsf{N}\},\qquad(6.7)$$

$$\mathcal{S}_{\mathsf{N}+1,b}^{(-,t,t')} \equiv (\eta_n)^{2(b-1)-\mathsf{N}}.\qquad(6.8)$$

Proof. We can compute the action of σ_n^-, by using the following reconstruction:

$$\sigma_n^- = \prod_{b=1}^{n-1} \bar{\mathsf{T}}(\eta_b) \mathsf{D}(\eta_n) \prod_{b=1}^{n} \frac{\bar{\mathsf{T}}(\eta_b/q)}{\det \bar{\mathsf{M}}(\eta_b)}, \tag{6.9}$$

so it holds:

$$\langle t|\sigma_n^-|t'\rangle = \frac{\prod_{h=1}^{n-1} t(\eta_h) \prod_{h=1}^{n} t'(\eta_h/q)}{\prod_{h=1}^{n} a(\eta_h) d(\eta_h/q)} \langle t|\mathsf{D}(\eta_n)|t'\rangle. \tag{6.10}$$

Now from the right D-SOV representation, we have:

$$\mathsf{D}(\eta_n)|t'\rangle = \sum_{\underbrace{h_1,\ldots,h_{\mathsf{N}}=0}_{h_n \text{ is missing}}}^{1} \frac{Q_{t'}(\eta_n/q)}{\omega(\eta_n/q)}(q - \frac{1}{q}) \prod_{a\neq n,a=1}^{\mathsf{N}} \left[(\frac{\eta_n q^{h_a}}{\eta_a} - \frac{\eta_a}{\eta_n q^{h_a}}) \frac{Q_{t'}(\eta_a q^{-h_a})}{\omega(\eta_a q^{-h_a})} \right]$$

$$\times \prod_{1\leq b<a\leq\mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b})|h_1,\ldots,h_n = 1,\ldots,h_{\mathsf{N}}\rangle, \tag{6.11}$$

and we can rewrite the coefficient as:

$$\frac{Q_{t'}(\eta_n/q)\,q}{\omega(\eta_n/q)\eta_n^{\mathsf{N}+1}} \prod_{a\neq n,a=1}^{\mathsf{N}} \frac{Q_{t'}(\eta_a q^{-h_a})q^{h_a}}{\omega(\eta_a q^{-h_a})\eta_a} V(\eta_1^2/q^{2h_1},\ldots,\eta_n^2/q^2,\ldots,\eta_{\mathsf{N}}^2/q^{2h_{\mathsf{N}}},\eta_n^2), \tag{6.12}$$

where $V()$ is the determinant of the $(\mathsf{N}+1) \times (\mathsf{N}+1)$ Vandermonde matrix in $\eta_1^2/q^{2h_1},\ldots,\eta_n^2/q^2,\ldots,\eta_{\mathsf{N}}^2/q^{2h_{\mathsf{N}}},\eta_n^2$. Now resumming as for the scalar product formula we derive our result. We have just to notice that the determinant which has in the line n the $\eta_n^{2(b-1)}$ gives zero and so we can add it to derive our result. $\square$

Theorem 6.2. *Let $\langle t|$ and $|t'\rangle$ be a left and a right eigenstate of the transfer matrix $\bar{\mathsf{T}}(\lambda)$, respectively, then it holds:*

$$\langle t|\sigma_n^z|t'\rangle = \frac{\prod_{h=1}^{n-1} t(\eta_h) \prod_{h=1}^{n} t'(\eta_h/q)}{-\prod_{h=1}^{n} a(\eta_h) d(\eta_h/q)} \det_{\mathsf{N}+1}(||\mathcal{S}_{a,b}^{(z,t,t')}||) \tag{6.13}$$

where $||\mathcal{S}_{a,b}^{(z,t,t')}||$ is the $(\mathsf{N}+1) \times (\mathsf{N}+1)$ matrix:

$$\mathcal{S}_{a,b}^{(z,t,t')} \equiv \Phi_{a,b}^{(t,t')} \text{ for } a \in \{1,\ldots,\mathsf{N}\}, \quad b \in \{1,\ldots,\mathsf{N}\} \tag{6.14}$$

$$\mathcal{S}_{\mathsf{N}+1,b}^{(z,t,t')} \equiv \eta_n^{2(b-1/2)-\mathsf{N}} \quad \text{for } b \in \{1,\ldots,\mathsf{N}\} \tag{6.15}$$

$$\mathcal{S}_{a,\mathsf{N}+1}^{(z,t,t')} \equiv \frac{Q_{t'}(\eta_a/q)\bar{Q}_t(\eta_a)}{\omega(\eta_a/q)} \left(\frac{\eta_a}{q}\right)^{\mathsf{N}-1} d(\eta_a/q) \text{ for } a \in \{1,\ldots,\mathsf{N}\}, \tag{6.16}$$

$$\mathcal{S}_{\mathsf{N}+1,\mathsf{N}+1}^{(z,t,t')} \equiv 1/2. \tag{6.17}$$

Proof. We can compute the action of σ_n^z, by using the following reconstruction:

$$\sigma_n^z = \prod_{b=1}^{n-1} \bar{\mathsf{T}}(\eta_b) \frac{\mathsf{C}(\eta_n) - \mathsf{B}(\eta_n)}{2} \prod_{b=1}^{n} \frac{\bar{\mathsf{T}}(\eta_b/q)}{\det \bar{\mathsf{M}}(\eta_b)}, \tag{6.18}$$

so it holds:

$$\langle t|\sigma_n^z|t'\rangle = \frac{\prod_{h=1}^{n-1} t(\eta_h) \prod_{h=1}^{n} t'(\eta_h/q)}{\prod_{h=1}^{n} a(\eta_h) d(\eta_h/q)} \langle t|\mathsf{C}(\eta_n)|t'\rangle - \frac{\langle t|t'\rangle}{2}. \tag{6.19}$$

19

Now from the right D-SOV representation of $\mathsf{C}(\eta_n)$, we have:

$$\mathsf{C}(\eta_n)|t'\rangle = \sum_{a=1}^{\mathsf{N}} \underbrace{\sum_{h_1,...,h_{\mathsf{N}}=0}^{1}}_{h_n \text{ is missing}} \left\{ \prod_{b\neq a,b=1}^{\mathsf{N}} \left[\frac{(\eta_n^2 - \eta_b^2 q^{-2h_b})}{(\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b})} \frac{Q_{t'}(\eta_b q^{-h_b})}{\omega_b(\eta_b q^{-h_b})} \right] \right.$$

$$\times \prod_{1\leq b<a\leq\mathsf{N}} (\eta_a^2 q^{-2h_a} - \eta_b^2 q^{-2h_b}) \frac{\eta_a^{\mathsf{N}-1} Q_{t'}(\eta_a q^{-h_a}) d(\eta_a q^{-h_a})}{\eta_n^{\mathsf{N}-1} q^{(\mathsf{N}-1)h_a} \omega(\eta_a q^{-h_a})} \Bigg\}_{h_n=1}$$

$$\times |h_1,...,h_n = 1 - \delta_{a,n},...,h_a - 1,...,h_{\mathsf{N}}\rangle, \qquad (6.20)$$

and so we can write:

$$\langle t|\mathsf{C}(\eta_n)|t'\rangle = \sum_{a=1}^{\mathsf{N}} (-1)^{\mathsf{N}+a} \underbrace{\sum_{h_1,...,h_{\mathsf{N}}=0}^{1}}_{h_n \text{ is missing}} \left\{ \widehat{V}_a(\eta_1^2/q^{2h_1},....,\eta_{\mathsf{N}}^2/q^{2h_{\mathsf{N}}},\eta_n^2) \right.$$

$$\times \prod_{b\neq a,b=1}^{\mathsf{N}} \frac{Q_{t'}(\eta_b q^{-h_b})\bar{Q}_t(\eta_b q^{-h_b})}{\omega(\eta_b q^{-h_b})} \frac{\eta_a^{\mathsf{N}-1} Q_{t'}(\eta_a q^{-h_a})\bar{Q}_t(\eta_a q^{1-h_a}) d(\eta_a q^{-h_a})}{\eta_n^{\mathsf{N}-1} q^{(\mathsf{N}-1)h_a} \omega(\eta_a q^{-h_a})} \Bigg\}_{h_n=1} \qquad (6.21)$$

In the last sum we can reintroduce the sum over h_n; indeed, $h_n = 0$ gives zero. Now, it is trivial to remark that the above sum minus $\langle t|t'\rangle/2$ is the develop of the determinant of the $(\mathsf{N}+1) \times (\mathsf{N}+1)$ matrix $||\mathcal{S}_{a,b}^{(z,t,t')}||$ presented in the statement of proposition. $\qquad\square$

7 Conclusion and outlooks

7.1 Results and first prospectives

In this article we have considered the spin 1/2 highest weight representations for the 6-vertex Yang-Baxter algebra on a generic N-sites finite lattice and analyzed the integrable quantum models associated to the antiperiodic transfer matrix. For this integrable quantum models, which in the homogeneous limit reproduce the XXZ spin 1/2 quantum chain with antiperiodic boundary conditions, we have obtained the following results:

- Complete characterization of the transfer matrix spectrum (eigenvalues/eigenstates) by separation of variables and proof of its simplicity.

- Reconstruction of all local operators in terms of the standard Sklyanin's quantum separate variables.

- Scalar Products: One determinant formulae of N×N matrices whose matrix elements are sums over the spectrum of each quantum separate variable of the product of the coefficients of states; for all the left/right separate states in the SOV-basis.

- Form factors of the local spin operators on the transfer matrix eigenstates in determinant form.

In the papers [102] and [103], the list of fundamental matrix elements for the antiperiodic XXZ spin-1/2 chain is completed with the the computation of the matrix elements on the transfer matrix eigenstates of the so-called density matrix and the two point functions.

Let us complete this subsection mentioning some facts to point out the relevance of the current results about form factors of local operators. First of all by using the decomposition of the identity (6.3) we can write any correlation function in spectral series of form factors. Then it is natural to expect that the correlation functions can be analyzed numerically mainly by the same tools developed in [111] in the ABA framework and used in the series of works[20] [111]-[117]. Indeed, also in our SOV framework we have determinant representations of the form factors and complete characterization of the transfer matrix spectrum in terms of the solutions of a system of Bethe equations. Finally, let us mention the important progresses[21] achieved recently [121]-[131] in computing the asymptotic behavior of correlation functions which are in principle susceptible to be extended to any (integrable) quantum model possessing determinant representations for the form factors of local operators [130] and so also to the models analyzed by our approach in the SOV framework.

7.2   Comparison with other SOV-type results

In the literature of quantum integrable models there exist results on the matrix elements of local operators which can be traced back to some applications of separation of variable methods. In this section, we try to recall those that we consider more relevant for us also as they allow an explicit comparison with our results leading to show an universal picture emerging in the characterization of matrix elements by SOV-methods.

In the case of the quantum integrable Toda chain [22], Smirnov [95] has derived in Sklyanin's SOV framework determinant formulae for the matrix elements of a conjectured basis of local operators which look very similar to our formulae. The main difference is due to the different nature of the spectrum of the quantum separate variables in these two models. In fact, in the case of the lattice Toda model Sklyanin's measure is continue (continuum SOV-spectrum) while it is discrete in our case. The elements of the matrices whose determinants give the form factor formulae are then expressed as "convolutions", over the spectrum of the separate variables, of Baxter equations solutions plus contributions coming from the local operators. In the case of the Smirnov's formulae they are true integral being the SOV-spectrum continuum while in our formulae they are "discrete convolutions" being the SOV-spectrum discrete. Let us comment that the need to conjecture[22] the form of a basis of local operators in [95] is due to the lack of a direct reconstruction of local operators in terms of Sklyanin's separate variables[23].

Even if in the different methodological contest of the S-matrix formulation of IQFTs, it is worth mentioning that the form factors of local operators [132] of the infinite volume quantum sine-Gordon field theory have also a form similar to the one predicted by SOV. This similarity statement can be made explicit considering for example the n-soliton form factors for the chiral local operators in the restricted sine-Gordon at the reflectionless points, $\beta^2 = 1/(1 + \nu)$ with $\nu \in \mathbb{Z}^{\geq 0}$; see formula (31) of [134]. Then, for some local fields (interpreted as primary operators in [134]) the corresponding form factors can be easily rewritten as determinants of n×n matrices whose elements are integral convolution of n-soliton wave functions (see the ψ-functions (32) of [134]) plus contributions coming from the local operators. The connection of these results with SOV emerges somehow naturally in [134] as the form factors of the quantum theory are there identified semi-classically by using the classical SOV reconstruction of the local fields [134].

---

[20]It is worth mentioning that important physical observables like the dynamical structure factors, accessible by neutron scattering experiments [104]-[110], were evaluated by this numerical approach.

[21]Results on asymptotic behaviour which has been also compared with previous more technical achievements rely mainly on the Riemann-Hilbert analysis of related Fredholm determinants [118]-[120].

[22]The consistence of this conjecture is there verified by a counting arguments based on the existence of an appropriate set of null conditions for the "integral convolutions".

[23]It is then worth citing that simple form of reconstructions of local operators in the quantum Toda model have been achieved by Babelon in [133] in terms of a set of quantum separate variables defined by a change of variables in terms of the original Sklyanin's quantum variables.

Finally, about representation of form factors in determinant form, it is worth pointing out the important achievements obtained recently in the series of works [135]-[143] where a fermionic basis of quasi-local operators has been introduced. There the expectation values of products of these operators on an appropriate vacuum state are written in determinant forms, similarly to the free fermions case by Wick's theorem. In particular, in [143] these results on form factors has been connected to the form factors analysis made in the S-matrix formulation for the restricted sine-Gordon at the reflectionless points made in [134, 144] where as above reported a link to SOV was made.

7.3   Outlook

Here, we want to point out the potential generality of the approach introduced to compute matrix elements of local operators for quantum integrable models. The main ingredients used to develop it are the solution of the transfer matrix spectral problem by SOV construction, the reconstruction of local operators in terms of the quantum separate variables and the scalar product formulae for the transfer matrix eigenstates (and general separate states). Then this SOV reconstruction (inverse problem solution) allows to write the action of any local operator on transfer matrix eigenstates as a finite sum of separate states in the SOV-basis. So that the matrix elements of any local operator are written as a finite sum of determinants of the scalar product formulae. The emerging picture is the possibility to apply this method to a whole class of integrable quantum models which were not tractable with other methods and in particular by algebraic Bethe ansatz.

The main aim is to implement explicitly this approach for a set of key integrable quantum models providing determinant representations for the matrix elements of local operators. To achieve this goal is very important as on the one hand it leads to the solution of fundamental quantum models previously unsolved and on the other hand gives the possibility to develop the mathematical tools to face the same problem for more involved integrable quantum models. This program has been already realized for several integrable quantum models as we will summarize in the following. In the case of the cyclic representations of the 6-vertex Yang-Baxter algebra like the lattice sine-Gordon model and the τ_2-model (of special interest for the connection with chiral Potts model) the form factors of local operators have been derived in [53] and [54], respectively. In [83] this approach is developed for the higher spin representations of highest weight type of the rational 6-vertex Yang-Baxter algebra which in the homogeneous limit leads to the higher spin XXX antiperiodic quantum chain. There the SOV setup is implemented and the form factors of the local spin operators on the transfer matrix eigenstates are obtained in a determinant form. In [84] this approach is developed for the spin 1/2 representations of highest weight type of the reflection algebra [145]-[151] which in the homogeneous limit leads to the XXZ open spin 1/2 quantum chains in quite general non-diagonal boundary conditions. There the SOV setup is implemented and the matrix elements of some interesting quasi-local string of local operators are computed. Further matrix elements are computed in [85] and in [152] for the most general representations of rational type; the relevance of these findings in the framework of the non-equilibrium systems like the simple exclusion processes is also discussed there. In [153] the SOV setup of the spectral problem is implemented for the spin 1/2 representations of highest weight type of the dynamical 6-vertex Yang-Baxter algebra and consequently for the corresponding 8-vertex Yang-Baxter algebra representations. There moreover the scalar product formulae for these representations are derived in a determinant form which allows to derive determinant formulae for the form factors as it will be presented in [154].

7.3.1   Toward solution of quantum field theories by integrable microscopic formulation

We are interested also in the use of integrable quantum models as a tool for the exact and complete characterization of the spectrum and dynamics of quantum field theories (QFTs) going through integrable lattice discretization. Then,

it is worth recalling that in the light-cone approach [155]-[158] the transfer matrix of the periodic XXZ spin-1/2 quantum chain with alternating inhomogeneities allows to define an integrable lattice regularization of the massive Thirring model. The continuum limit and the infrared limit defining the massive Thirring QFT in the finite volume and the infinite volume, respectively, have been implemented and their spectrum analyzed in the series of papers [159]-[166]. Then in the infinite volume limit, where the boundary conditions of the original spin chain should not play a role, we have the possibility to describe the massive Thirring QFT by using as starting point the SOV approach of the present paper which on the lattice is known to give a complete characterization of the spectrum. The analysis of this interesting issue by the implementation of the required limits and the comparison with the known results [159]-[166], derived instead in the framework of the algebraic Bethe ansatz, will be the subject of a forthcoming publication.

Finally, let us point out that our interest in integrable lattice regularizations of QFTs is due to the possibility to define an exact setup where to use the reconstruction of local fields in terms of the quantum separate variables to overcome the longstanding problem of their identifications in the S-matrix formulation[24]. Different methods have been introduced to address this problem and one important line of research is related to the description of massive integrable quantum field theories (IQFTs) as (superrenormalizable) perturbations of conformal field theories [172]-[176] by relevant local fields [177]-[180]. This characterization has led to the expectation that the perturbations do not change the structure of the local fields in this way leading to the attempt to classify the local field content of massive theories[25] by that of the corresponding ultraviolet conformal field theories. Several results are known which confirm this characterization; see for example [196]-[199] for the proof of the isomorphism restricted to the chiral sector of some IQFTs[26] and the series of works [207]-[210] where the first rigorous proof of the isomorphism for the full operator space was given for the massive Lee-Yang model. However, while these are important results on the global structure of the operator space in the S-matrix formulation of the massive IQFTs they do not really lead to the identification of particular local fields[27] which then remain the main missing information in the S-matrix formulation.

---

[24]See [167]-[171] for a review and references therein.

[25]In the S-matrix formulation the local fields are characterized in terms of form factors (matrix elements on the asymptotic particle states) and many results are known on these form factors in IQFTs; see for example [181]-[195] and references therein.

[26]An important role in these studies has been played by the fermionic representations of the characters, as derived for different classes of rational conformal field theories in [200]-[206].

[27]A part for some local fields, like the components of the stress energy tensor, which can be characterized by physical prescription [207] and [209].

8 Appendix

In this appendix we report the SOV-representations of the Yang-Baxter generators in the right and left eigenbasis of $\mathsf{A}(\lambda)$.

Theorem 8.1. *l) Left $\mathsf{A}(\lambda)$ SOV-representations: Under the condition (3.1), the states:*

$$\langle h_1, ..., h_\mathsf{N}|| \equiv \frac{1}{\mathsf{N}} \langle 0| \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathsf{C}(\eta_n/q)}{d(\eta_n/q)} \right)^{h_n}, \qquad (8.1)$$

where $h_n \in \{0, 1\}$, $n \in \{1, ..., \mathsf{N}\}$, define a $\mathsf{A}(\lambda)$-eigenbasis of $\mathcal{L}_\mathsf{N}$:

$$\langle h_1, ..., h_\mathsf{N}||\mathsf{A}(\lambda) = a_{\boldsymbol{h}}(\lambda)\langle h_1, ..., h_\mathsf{N}||, \qquad (8.2)$$

where:

$$d_{\boldsymbol{h}}(\lambda) \equiv \prod_{n=1}^{\mathsf{N}} \left(\frac{\lambda q^{(1-h_n)}}{\eta_n} - \frac{\eta_n}{\lambda q^{(1-h_n)}} \right) \quad \text{and} \quad \boldsymbol{h} \equiv (h_1, ..., h_\mathsf{N}). \qquad (8.3)$$

The action of the remaining Yang-Baxter generators on the generic state $\langle h_1, ..., h_\mathsf{N}||$ reads:

$$\langle h_1, ..., h_\mathsf{N}||\mathsf{C}(\lambda) = \sum_{a=1}^{\mathsf{N}} \prod_{b \neq a} \frac{\lambda q^{(1-h_b)}/\eta_b - \eta_b/q^{(1-h_b)}\lambda}{\eta_a q^{(h_a-h_b)}/\eta_b - \eta_b/q^{(h_a-h_b)}\eta_a} d(\eta_a q^{h_a-1})\langle h_1, ..., h_\mathsf{N}||T_a^+, \qquad (8.4)$$

$$\langle h_1, ..., h_\mathsf{N}||\mathsf{B}(\lambda) = \sum_{a=1}^{\mathsf{N}} \prod_{b \neq a} \frac{\lambda q^{(1-h_b)}/\eta_b - \eta_b/q^{(1-h_b)}\lambda}{\eta_a q^{(h_a-h_b)}/\eta_b - \eta_b/q^{(h_a-h_b)}\eta_a} a(\eta_a q^{h_a-1})\langle h_1, ..., h_\mathsf{N}||T_a^-. \qquad (8.5)$$

where:

$$\langle h_1, ..., h_a, ..., h_\mathsf{N}||T_a^{\pm} = \langle h_1, ..., h_a \pm 1, ..., h_\mathsf{N}||. \qquad (8.6)$$

Finally, $\mathsf{D}(\lambda)$ is uniquely defined by the quantum determinant relation.

II) Right $\mathsf{A}(\lambda)$ SOV-representations: Under the condition (3.1), the states:

$$||h_1, ..., h_\mathsf{N}\rangle \equiv \frac{1}{\mathsf{N}} \prod_{n=1}^{\mathsf{N}} \left(\frac{\mathsf{B}(\eta_n/q)}{a(\eta_n)} \right)^{h_n} |0\rangle, \qquad (8.7)$$

where $h_n \in \{0, 1\}$, $n \in \{1, ..., \mathsf{N}\}$, define a $\mathsf{A}(\lambda)$-eigenbasis of $\mathcal{R}_\mathsf{N}$:

$$\mathsf{A}(\lambda)||h_1, ..., h_\mathsf{N}\rangle = a_{\boldsymbol{h}}(\lambda)|h_1, ..., h_\mathsf{N}\rangle. \qquad (8.8)$$

The action of the remaining Yang-Baxter generators on the generic state $||h_1, ..., h_\mathsf{N}\rangle$ reads:

$$\mathsf{C}(\lambda)||h_1, ..., h_\mathsf{N}\rangle = \sum_{a=1}^{\mathsf{N}} T_a^- ||h_1, ..., h_\mathsf{N}\rangle \prod_{b \neq a} \frac{\lambda q^{(1-h_b)}/\eta_b - \eta_b/q^{(1-h_b)}\lambda}{\eta_a q^{(h_a-h_b)}/\eta_b - \eta_b/q^{(h_a-h_b)}\eta_a} d(\eta_a q^{-h_a}), \qquad (8.9)$$

$$\mathsf{B}(\lambda)||h_1, ..., h_\mathsf{N}\rangle = \sum_{a=1}^{\mathsf{N}} T_a^+ ||h_1, ..., h_\mathsf{N}\rangle \prod_{b \neq a} \frac{\lambda q^{(1-h_b)}/\eta_b - \eta_b/q^{(1-h_b)}\lambda}{\eta_a q^{(h_a-h_b)}/\eta_b - \eta_b/q^{(h_a-h_b)}\eta_a} a(\eta_a q^{-h_a}). \qquad (8.10)$$

where:

$$T_a^{\pm}||h_1, ..., h_a, ..., h_\mathsf{N}\rangle = ||h_1, ..., h_a \pm 1, ..., h_\mathsf{N}\rangle. \qquad (8.11)$$

Finally, $\mathsf{D}(\lambda)$ is uniquely defined by the quantum determinant relation.

Note that the proof of this theorem can be given along the same lines used to prove the Theorem 3.2.

References

[1] C. Itzykson and J. M. Drouffe, Statistical Field Theory, Cambridge University Press, (1991).

[2] B. McCoy, *The connection between statistical mechanics and quantum field theory*, in Statistical Mechanics and Field Theory, eds. V.V. Bazhanov and C.J. Burden, World Scientific Singapore (1995) 26.

[3] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York U.S.A. (1982).

[4] R. Dashen, S. K. Ma and H. J. Bernstein, Phys. Rev. **187** (1969) 345.

[5] C. N. Yang, Phys. Rev. Lett. **19** (1967) 1312.

[6] C. N. Yang and C. P. Yang, J. Math. Phys. **10** (1969) 1115.

[7] Al. B. Zamolodchikov, Nucl. Phys. B **342** (1990) 695.

[8] E. K. Sklyanin and L. D. Faddeev, Sov. Phys. Dokl. **23** (1978) 902.

[9] E. K. Sklyanin and L. A. Takhtajan, L.D. Faddeev, Theor. Math. Phys. **40** (1980) 688.

[10] L. D. Faddeev and L. A. Takhtajan, Russ. Math. Surveys, **34** : 5 (1979) 11.

[11] E. K. Sklyanin, Dokl. Akad. Nauk SSSR **244** (1979) 1337; Soviet Physics Dokl. **24** (1979) 107.

[12] P. P. Kulish and E. K. Sklyanin, Phys. Lett. A **70** (1979) 461.

[13] L. D. Faddeev, Sov. Sci. Rev. Math. Cl (1980) 107.

[14] E. K. Sklyanin, J. Sov. Math. **19** (1982) 1546.

[15] L. D. Faddeev, Les Houches lectures of 1982, Elsevier Sci. Publ. **563** (1984).

[16] L. D. Faddeev, *How Algebraic Bethe Ansatz works for integrable model*, hep-th/9605187v1.

[17] M. Jimbo, Adv. Series in Math. Phys. **10**, Singapore: World Scientific, (1990).

[18] P. P. Kulish and E. K. Sklyanin, Lect. Notes Phys. **151** (1982) 61.

[19] B. S. Shastry, Lect. Notes Phys, **242** (1985).

[20] H. B. Thacker, Rev. Mod. Phys. **53** (1981) 253.

[21] A. G. Izergin and V. E. Korepin, Nucl. Phys. B **205** (1982) 401.

[22] E. K. Sklyanin, Lect. Notes Phys. **226** (1985) 196; E. K. Sklyanin J. Sov. Math. **31** (1985) 3417.

[23] E. K. Sklyanin, *Quantum inverse scattering method. Selected topics*. In: *Quantum groups and quantum integrable systems* (World Scientific, 1992) 63–97, arXiv:hep-th/9211111v1.

[24] E. K. Sklyanin, Prog. Theor. Phys. Suppl. **118** (1995) 35.

[25] N. Kitanine, J. Phys. A: Math. Gen. **34** (2001) 8151.

[26] O. A. Castro-Alvaredo, J. M. Maillet, J. Phys. A **40** (2007) 7451.

[27] N. Kitanine, K.K. Kozlowski, J.M. Maillet, G. Niccoli, N.A. Slavnov, V. Terras , J. Stat. Mech. (2007) P10009.

[28] K. K. Kozlowski, J.Stat.Mech. (2008) P02006.

[29] N. Kitanine, K. Kozlowski, J. M. Maillet, G. Niccoli, N. A. Slavnov, V. Terras, J. Stat. Mech. (2008) P07010.

[30] N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B **554** (1999) 647.

[31] J. M. Maillet and V. Terras, Nucl. Phys. B **575** (2000) 627.

[32] R. J. Baxter, J. Stat. Phys. **117** (2004) 1.

[33] G. Niccoli and J. Teschner, J. Stat. Mech. (2010) P09014.

[34] G. Niccoli, Nucl. Phys. B **835** (2010) 263.

[35] G. Niccoli, JHEP JHEP03(2011)123.

[36] G. von Gehlen, N. Iorgov, S. Pakuliak and V. Shadura, J. Phys. A: Math. Gen. **39** (2006) 7257.

[37] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura and Yu Tykhyy, J. Phys. A: Math. Theor. **40** (2007) 14117.

[38] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura and Yu Tykhyy, J. Phys. A: Math. Theor. **41** (2008) 095003.

[39] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura, J. Phys. A: Math. Theor. **42** (2009) 304026.

[40] N. Grosjean and G. Niccoli, *The τ_2-model and the chiral Potts model revisited: Completeness of Bethe equations originated from Sklyanin SOV*, to appear.

[41] V. V. Bazhanov, Y. G. Stroganov, J. Stat. Phys. **59** (1990) 799.

[42] R. J. Baxter, V. V. Bazhanov and J. H. H. Perk, Int. J. Mod. Phys. **B4** (1990) 803.

[43] R. J. Baxter, J. Stat. Phys. **57** (1989) 1.

[44] G. Albertini, B. M. McCoy and J. H. H. Perk, Adv. Study in Pure Math. **19** (1989) 1.

[45] G. Albertini, B. M. McCoy and J. H. H. Perk, Phys. Lett. A **135** (1989) 159.

[46] G. Albertini, B. M. McCoy and J. H. H. Perk, Phys. Lett. A **139** (1989) 204.

[47] R. J. Baxter, J. H. H. Perk and H. Au-Yang, Phys. Lett. A **128** (1988) 138; H. Au-Yang and J. H. H. Perk, Adv. Studies Pure Math. **19** (1989).

[48] R. J. Baxter, Phys. Lett. A **133** (1989) 185.

[49] H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. Yan, Phys. Lett. A **123** (1987) 219.

[50] B. M. McCoy, J. H. H. Perk, S. Tang, and C. H. Sah, Phys. Lett. A **125** (1987) 9.

[51] H. Au-Yang, B. M. McCoy, J. H. H. Perk, and S. Tang, Algebraic Analysis, Vol. 1, M. Kashiwara and T. Kawai, eds (Academic Press, New York, 1988).

[52] V. O. Tarasov Phys. Lett. A **147** (1990) 487.

[53] N. Grosjean, J. M. Maillet, G. Niccoli, *On the form factors of local operators in the lattice sine-Gordon model*, arXiv:1204.6307.

[54] N. Grosjean, J. M. Maillet, G. Niccoli, *On the form factors of local operators in the τ_2-model and the chiral Potts model*, to appear.

[55] W. Heisenberg, Z. Phys. **49** (1928) 619.

[56] H. Bethe, Z. Phys. **71** (1931) 205.

[57] L. Hulthen, Ark. Mat. Astron. Fys. **26** (1938) 1.

[58] R. Orbach, Phys. Rev. **112** (1958) 309.

[59] L. R. Walker, Phys. Rev. **116** (1959) 1089.

[60] C. N. Yang and C. P. Yang, Phys. Rev. **150** (1966) 321.

[61] C. N. Yang and C. P. Yang, Phys. Rev. **150** (1966) 327.

[62] M. Gaudin, La Fonction d'onde de Bethe Paris, Masson, 1983.

[63] E. H. Lieb and D. C. Mattis, Mathematical Physics in One Dimension, New-York Academic, 1966.

[64] A. G. Izergin, N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B **554** (1999) 679.

[65] N. Kitanine, J. M. Maillet, and V. Terras, Nucl. Phys. B **567** (2000) 554.

[66] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B **641** (2002) 487.

[67] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B **642** (2002) 433.

[68] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A **35** (2002) L385.

[69] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A **35** (2002) L753.

[70] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B **712** (2005) 600.

[71] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, Nucl. Phys. B **729** (2005) 558.

[72] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Phys. A **38** (2005) 7441.

[73] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, J. Stat. Mech. (2005) L09002.

[74] N. Kitanine, J. M. Maillet, N. A. Slavnov, and V. Terras, *On the algebraic Bethe Ansatz approach to the correlation functions of the XXZ spin-1/2 Heisenberg chain*, In *Recent Progress in Solvable lattice Models*, RIMS Sciences Project Research 2004 on *Method of Algebraic Analysis in Integrable Systems*, RIMS, Kyoto, Kokyuroku, **1480**, 14 (2006); hep-th/0505006.

[75] N. Kitanine, K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. (2007) P01022.

[76] C. M. Yung, M. T. Batchelor, Nucl. Phys. B **446** (1995) 461.

[77] M. T. Batchelor, R. J. Baxter, M. J. O'Rourke, C. M. Yung, J. Phys. A **28** (1995) 2759.

[78] S. Niekamp, T. Wirth, H. Frahm, J. Phys. A **42** (2009) 195008.

[79] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter and G. R. W. Quispel, J. Phys. A **20** (1987) 6397.

[80] N. Y. Reshetikhin, Lett. Math. Phys. **7** (1983) 205.

[81] N. Y. Reshetikhin, Sov. Phys. JETP **57** (1983) 691.

[82] E. Mukhin, V. Tarasov and A. Varchenko, Comm. Math. Phys. **228** (2009) 1.

[83] G. Niccoli, *Antiperiodic higher spin XXX quantum chains by separation of variables: Form factors and complete spectrum*, to appear.

[84] G. Niccoli, *Non-diagonal open spin-1/2 XXZ quantum chain by separation of variables: Complete spectrum and matrix elements of some quasi-local operators*, to appear.

[85] G. Niccoli, *Antiperiodic dynamical 6-vertex and periodic 8-vertex models: Scalar products and complete spectrum by separation of variables*, to appear.

[86] A. G. Izergin and V. E. Korepin, Doklady Akademii Nauk **259** (1981) 76.

[87] A. G. Izergin and V. E. Korepin, *A lattice model related to the nonlinear Schroedinger equation*, arXiv:0910.0295

[88] J. M. Maillet, and J. Sanchez de Santos, Amer. Math. Soc. Transl. **201** (2000) 137.

[89] V. G. Drinfel'd, Soviet Math. Dokl. **28** (1983) 667.

[90] V. G. Drinfel'd, *Proceedings of the the International Congress of Mathematicians*, Berkeley, USA, 1986, 798-820, AMS, 1987.

[91] V. G. Drinfel'd, Leningrad Math. J. **1** (1990) 1419.

[92] V. Terras, Lett. Math. Phys. **48** (1999) 263.

[93] N. A. Slavnov, Theor. Math. Phys. **79** (1989) 502.

[94] N. A. Slavnov, Zap. Nauchn. Semin. POMI **245** (1997) 270.

[95] F. Smirnov, J. Phys. A: Math. Gen. **31** (1998) 8953.

[96] A. Bytsko, J. Teschner, J. Phys. **A39** (2006) 12927.

[97] P. P. Kulish, N. Y. Reshetikhin and E. K. Sklyanin, Lett. Math. Phys. **5** (1981) 393.

[98] A. N. Kirillov and N. Y. Reshetikhin, J. Phys. A: Math Gen. **20** (1987) 1565.

[99] V. V. Bazhanov and N. Y. Reshetikhin, Int. J. Mod. Phys. A **4** (1989) 115.

[100] R. I. Nepomechie, Nucl. Phys. B **622** (2002) 615.

[101] R. I. Nepomechie, J. Stat. Phys. **111** (2003) 1363.

[102] N. Grosjean and G. Niccoli, *Matrix elements of the density matrix in antiperiodic spin-1/2 XXZ quantum chain by separation of variables*, to appear.

[103] N. Grosjean and G. Niccoli, *Two-point correlation functions in antiperiodic spin-1/2 XXZ quantum chain by separation of variables*, to appear.

[104] F. Bloch, Phys. Rev. **50** (1936) 259.

[105] J. S. Schwinger, Phys. Rev. **51** (1937) 544.

[106] O. Halpern and M. H. Johnson, Phys. Rev. **55** (1938) 898.

[107] L. Van Hove, Phys. Rev. **95** (1954) 249.

[108] L. Van Hove, Phys. Rev. **95** (1954) 1374.

[109] Marshall W and Lovesey S W, *Theory of Thermal Neutron Scattering*, Oxford, Academic (1971).

[110] R. Balescu, *Equilibrium and Nonequilibrium Statistical Mechanics*, New York: Wiley (1975).

[111] J.-S. Caux, J. M. Maillet, Phys. Rev. Lett. **95** (2005) 077201.

[112] J.-S. Caux, R. Hagemans, J. M. Maillet, J. Stat. Mech. (2005) P09003.

[113] R. G. Pereira, J. Sirker, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, Phys. Rev. Lett. **96** (2006) 257202.

[114] R. Hagemans, J.-S. Caux, J. M. Maillet, Proceedings of the "Tenth Training Course in the Physics of Correlated Electron Systems and High-Tc Superconductors", Salerno, Oct 2005, AIP Conference Proceedings 846 (2006) 245.

[115] R. G. Pereira, J. Sirker, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, J. Stat. Mech. (2007) P08022.

[116] J. Sirker, R. G. Pereira, J.-S. Caux, R. Hagemans, J. M. Maillet, S. R. White, I. Affleck, Proceedings SCES '07, Houston, Physica B **403** (2008) 1520.

[117] J.-S. Caux, P. Calabrese and N. A. Slavnov, J. Stat. Mech. Theory Exp. (2007) P01008.

[118] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, Com. Math. Phys. **291** (2009) 691.

[119] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Stat. Mech. (2009) P04003.

[120] K. K. Kozlowski, *Riemann–Hilbert approach to the time-dependent generalized sine kernel*, arXiv:1011.5897.

[121] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, J. Math. Phys. **50** (2009) 095209 .

[122] K. K. Kozlowski, J. Math. Phys. **50** (2009) 095205.

[123] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, *Long-distance behavior of temperature correlation functions in the one-dimensional Bose gas*, arXiv:1003.4557.

[124] K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, J.Stat.Mech.(2011) P03018.

[125] K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, J.Stat.Mech.(2011) P03019.

[126] K. K. Kozlowski, *Low-T asymptotic expansion of the solution to the Yang-Yang equation*, arXiv:1112.6199.

[127] K. K. Kozlowski, *On Form Factors of the conjugated field in the non-linear Schrödinger model*, arXiv:1105.1052.

[128] K. K. Kozlowski, *Large-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model*, arXiv:1101.1626.

[129] K. K. Kozlowski, V. Terras, *Long-time and large-distance asymptotic behavior of the current-current correlators in the non-linear Schrödinger model*, arXiv:1101.0844.

[130] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov, V. Terras, *Form factor approach to the asymptotic behavior of correlation functions in critical models*, arXiv:1110.0803.

[131] K. K. Kozlowski, B. Pozsgay, *Surface free energy of the open XXZ spin-1/2 chain*, arXiv:1201.5884

[132] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, World Scientific, (1992).

[133] O. Babelon, J. Phys. A **37** (2004) 303.

[134] O. Babelon, D. Bernard, F. Smirnov, Comm. Math. Phys. **182** (1996) 319.

[135] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, and Y. Takeyama, Com. Math. Phys. **272** (2007) 263.

[136] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, Y. Takeyama, *Fermionic basis for space of operators in the XXZ model*, arXiv:hep-th/0702086.

[137] M. Jimbo, T. Miwa, F. Smirnov, *On one-point functions of descendants in sine-Gordon model*, arXiv:0912.0934.

[138] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, Y. Takeyama, Comm. Math. Phys. **286** (2009) 875.

[139] M. Jimbo, T. Miwa, F. Smirnov, J. Phys. A **42** (2009) 304018.

[140] H. Boos, M. Jimbo, T. Miwa, F. Smirnov, *Hidden Grassmann Structure in the XXZ Model IV: CFT limit*, arXiv:0911.3731.

[141] M. Jimbo, T. Miwa, F. Smirnov, Nucl. Phys. B **852** (2011) 390.

[142] M. Jimbo, T. Miwa, F. Smirnov, *Fermionic screening operators in the sine-Gordon model*, arXiv:1103.1534.

[143] M. Jimbo, T. Miwa, F. Smirnov, Lett. Math. Phys. **96** (2011) 325.

[144] O. Babelon, D. Bernard, F. Smirnov, Comm. Math. Phys. **186** (1997) 601.

[145] E. K. Sklyanin, J. Phys. A: Math. Gen. **21** (1988) 2375.

[146] I. V. Cherednik, Theor. Math. Phys. **61** (1984) 977.

[147] P. P. Kulish and S. K. Sklyanin, J. Phys. A: Math. Gen. **24** (1991) L435.

[148] L. Mezincescu and R. Nepomechie, Int. J. Mod. Phys. A **6** (1991) 5231.

[149] P. P. Kulish and E. K. Sklyanin, J. Phys. A: Math. Gen. **25** (1992) 5963.

[150] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A **9** (1994) 3841.

[151] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A **9** (1994) 4353.

[152] N. Kitanine, G. Niccoli, *Quantum separation of variables approach to non-equilibrium problems I: Some physical observables of asymmetric simple exlusion processes*, to appear.

[153] N. Kitanine, G. Niccoli, *Quantum separation of variables approach to non-equilibrium problems II: Some physical observables of asymmetric simple exlusion processes*, to appear.

[154] D. Levy-Bencheton, G. Niccoli and V. Terras, *Antiperiodic dynamical 6-vertex and periodic 8-vertex models: Form factors by separation of variables*, to appear.

[155] C. Destri and H. J. de Vega, Nucl. Phys. B **290** (1987) 363.

[156] H. J. de Vega, Int. J. Mod. Phys. A (1989) 2371; Int. J. Mod. Phys. B (1990) 735.

[157] C. Destri and H. J. de Vega, J. Phys. A **22** (1989) 1329.

[158] C. Destri and H. J. de Vega, Nucl. Phys. B **406** (1993) 566.

[159] C. Destri, H. J. De Vega, Phys. Rev. Lett. **69** (1992) 2313.

[160] C. Destri and H. J. de Vega, Nucl.Phys. B **438** (1995) 413.

[161] C. Destri, H. J. de Vega Nucl. Phys. B **504** (1997) 621.

[162] D. Fioravanti, A. Mariottini, E. Quattrini, F. Ravanini, Phys. Lett. B **390** (1997) 243.

[163] G. Feverati, F. Ravanini, G. Takacs, Phys. Lett. B **430** (1998) 264.

[164] G. Feverati, F. Ravanini, G. Takacs, Nucl. Phys. B **540** (1999) 543.

[165] G. Feverati, *Finite volume spectrum of Sine-Gordon model and its restrictions*, Ph.D. Thesis, Bologna University, hep-th/0001172, 2000.

[166] F. Ravanini, *Finite size effects in integrable quantum field theories*, hep-th/0102148.

[167] A. B. Zamolodchikov, Pis. Zh. Eksp. Teor. Fiz. **25** (1977) 499; A. B. Zamolodchikov, Comm. Math. Phys. **55** (1977) 183 (Engl. transl.).

[168] A. B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. **120** (1979) 253.

[169] M. Karowski and H. J. Thun, Nucl. Phys. B **130** (1977) 295.

[170] V. E. Korepin, Comm. Math. Phys. **76** (1980) 165.

[171] G. Mussardo, Phys. Rep. **218** (1992) 215.

[172] M. A. Virasoro, Phys. Rev. D **1** (1970) 2933.

[173] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. B **241** (1984) 333.

[174] P. Ginsparg, "Applied Conformal Field Theory", in: Fields, Strings and Critical Phenomena, Les Houches Lecture Notes 1988, eds. E. Brézin and J. Zinn-Justin, Elsevier, New York (1989).

[175] J. L. Cardy, *Conformal Invariance and Statistical Mechanics*, in Fields, Strings and Critical Phenomena, ed. E. Brézin and J. Zinn-Justin, Les Houches (1988), Session XLIX North-Holland, Amsterdam (1990) 169.

[176] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer, New York (1997).

[177] A. B. Zamolodchikov, Int. J. Mod. Phys. A **3** (1988) 743.

[178] A. B. Zamolodchikov, Adv. Stud. Pure Math. **19** (1989) 641.

[179] Al. B. Zamolodchikov, Nucl. Phys. B **348** (1991) 619.

[180] R. Guida, N. Magnoli, Nucl. Phys. B **471** (1996) 361.

[181] M. Karowski, P. Weisz, Nucl. Phys. B **139** (1978) 455.

[182] F. A. Smirnov, Physica A **3** (1988) 743.

[183] F. A. Smirnov, Nucl. Phys. B **337** (1990) 156.

[184] A. Fring, G. Mussardo and P. Simonetti, Nucl. Phys. B **393** (1993) 413.

[185] A. Fring, G. Mussardo and P. Simonetti, Phys. Lett. B **307** (1993) 83.

[186] A. Koubek and G. Mussardo, Phys. Lett. B **311** (1993) 193.

[187] G. Mussardo and P. Simonetti, Int. J. Mod. Phys. A **9** (1994) 3307.

[188] A. Koubek, Nucl. Phys. B **428** (1994) 655.

[189] G. Delfino and G. Mussardo, Nucl. Phys. B **455** (1995) 724.

[190] G. Delfino, P. Simonetti and J. L. Cardy, Phys. Lett. B **387** (1996) 327.

[191] H. Babujian, A. Fring, M. Karowski and A. Zapletal, Nucl. Phys. B **538** (1999) 535.

[192] S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B **607** (2001) 437.

[193] H. Babujian and M. Karowski, Nucl. Phys. B **620** (2002) 407.

[194] H. Babujian and M. Karowski, J. Phys. A: Math. Gen. **35** (2002) 9081.

[195] G. Delfino, J. Phys. A: Math. Gen. **37** (2004) R45.

[196] J. L. Cardy and G. Mussardo, Nucl. Phys. B **340** (1990) 387.

[197] A. Koubek, Nucl. Phys. B **435** (1995) 703.

[198] F. Smirnov, Nucl. Phys. B **453** (1995) 807.

[199] M. Jimbo, T. Miwa, Y. Takeyama, *Counting minimal form factors of the restricted sine-Gordon model*, arXiv:math-ph/0303059v6.

[200] R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, Phys. Lett. B **304** (1993) 263.

[201] R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, Phys. Lett. B **307** (1993) 68.

[202] S. Dasmahapatra, R. Kedem, T. R. Klassen, B. M. McCoy, E. Melzer, J. Mod. Phys. B **7** (1993) 3617.

[203] B. L. Feigin, T. Nakanishi, H. Ooguri, Int. J. Mod. Phys. A **7** (Suppl. 1A) (1992) 217.

[204] W. Nahm, A. Recknagel, M. Terhoeven, Mod. Phys. Lett. A **8** (1993) 1835.

[205] R. Kedem, B. M. McCoy and E. Melzer, *The sums of Rogers, Schur and Ramanujan and the Bose-Fermi correspondence in 1+1 dimensional quantum field theory*, in : *Recent Progress in Statistical Mechanics and Quantum Field Theory*, ed. P. Bouwknegt et all., World Scientific, Singapore, 1995.

[206] A. Berkovich and B. M. McCoy, *Rogers-Ramanujan identities: A century of progress from mathematics to physics*, Proceedings of ICM98 Documenta Mathematica, (Bielefeld, Germany: Deutscher Mathematiker-Vereinigung) Extra volume ICM 1998, **3** (1998) 163.

[207] G. Delfino and G. Niccoli, Nucl. Phys. B**707** (2005) 381.

[208] G. Delfino and G. Niccoli, J. Stat. Mech. (2005) P04004

[209] G. Delfino and G. Niccoli, JHEP 05 (2006) 035.

[210] G. Delfino and G. Niccoli, Nucl. Phys. B **799** (2008) 364.